11th All Strands
03/12/2008

Student Name: ______________________________

Class: ______________________________

Date: ______________________________

Instructions: **Read each question carefully and select the correct answer.**

1. Solve the following system of equations using matrices.

$$-x - 3y + 5z = 23$$
$$3x + y + 2z = 15$$
$$2x + 2y + 4z = 18$$

A. x = 91/45, y = -22/45, z = 212/45
B. x = 68 1/8, y = -8 3/8, z = 4
C. x = -23, y = -21/2, z = 4
D. x = 3, y = -2, z = 4

2. Solve the equation.

$$\begin{bmatrix} \frac{1}{9} & \frac{4}{9} \\ \frac{6}{9} & \frac{8}{9} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$$

A. $x = \frac{1}{9}$ and $y = \frac{5}{9}$
B. $x = -36$ and $y = 18$
C. $x = \frac{1024}{729}$ and $y = \frac{512}{729}$
D. $x = -\frac{28}{9}$ and $y = -\frac{40}{9}$

3. What is the minimum value given the following constraints?

Objective Quantity: $C = 3x - 5y$
Constraints:
$x \geq 0$
$y \geq 0$
$3x - 2y \leq 12$
$5x + 2y \leq 20$

A. - 50
B. 12
C. - 30
D. No Minimum

4. What is the minimum value given the following constraints?

Objective Quantity: $C = 37x + 52y$
Constraints:
$x \leq 0$
$y \geq 0$
$6x - 5y \geq -300$
$3x + 8y \geq -24$

A. 3,120
B. - 856
C. - 296
D. No Minimum

5. Classify the conic.

$$-10x^2 + 7xy - 4y^2 + 10x + y = 0$$

A. Circle
B. Hyperbola
C. Parabola
D. Ellipse

6. Classify the conic.

$$2x^2 + 8xy + 3y^2 + 5x - 2y = 0$$

A. Circle
B. Hyperbola
C. Parabola
D. Ellipse

7. Distribute. Simplify your answer.

-6i(3 + 4i)

A. $-18i + 24i^2$

B. $24 - 18i$

C. $18 - 24i^2$

D. $-18i - 24$

8. Multiply. Simplify your answer.

(5 - 6i)(3 - 4i)

A. $24i^2 + 15$

B. $24i^2 - 38i + 15$

C. $-9 - 38i$

D. $-10i + 8$

9. A cube has the following vertices: (-2, 5, 4), (-2, 0, 4), and (3, 0, 4). Which point is also a vertex of the cube?

A. (-2, 4, 5)
B. (3, 0, 5)
C. (-2, 5, 0)
D. (3, 5, 9)

10. A cube has the following vertices: (4, 7, -1), (4, 6, -1), and (5, 6, -1). Which point is also a vertex of the cube?

A. (4, 7, 6)
B. (4, 6, 0)
C. (5, -1, 7)
D. (4, 0, 6)

11. Find the supplement of $\frac{8\pi}{17}$.

A. $\frac{26\pi}{17}$

B. $\frac{9\pi}{17}$

C. $\frac{8\pi}{17}$

D. $\frac{\pi}{17}$

12. Which of the following is a coterminal angle for $\theta = \frac{2\pi}{3}$?

A. $\frac{\pi}{3}$

B. $\frac{5\pi}{3}$

C. $\frac{8\pi}{3}$

D. $-\frac{2\pi}{3}$

13. Rewrite 320° in radians.

A. $\frac{16\pi}{9}$ radians

B. $\frac{9\pi}{16}$ radians

C. $\frac{2\pi}{9}$ radians

D. $\frac{9\pi}{2}$ radians

14. Find two angles that are coterminal.

A. 60° and 240°
B. 35° and -145°
C. 40° and -340°
D. 75° and 435°

15. Complete the following trigonometric identity.

$\frac{1}{\cot^2\theta} = \_\_\_\_\_$.

A. $\sin^2\theta$

B. $\tan^2\theta$

C. $\sec^2\theta$

D. $\cos^2\theta$

16. Choose the statement that is true.

A. $1-\sec^2(x) = 1-\tan^2(x)$

B. $\cot^2(x)\left[\sec^2(x)-1\right] = 1$

C. $\cos(x)\left(1+\tan^2(x)\right) = \sin(x)$

D. $\cos^2(x)-\sin^2(x) = 1+2\sin^2(x)$

17. Describe the left and right behaviors of the graph of the following function.

$$f(x) = x^6 + 4x^2 + 1$$

A.

f increases on $[-\infty, 0]$, and f decreases on $[0, \infty$

B.

f decreases on $[-\infty, 0]$, and f increases on $[0, \infty$

C.

f increases on $[-\infty, 0]$, and f increases on $[0, \infty]$

D.

f decreases on $[-\infty, 0]$, and f decreases on $[0, \infty$

18. State the maximum number of turns in the graph of the polynomial.
$f(x) = 3x^6 + 2x + 4$

A. 3 turns
B. 5 turns
C. 6 turns
D. 7 turns

19. Subtract the fractions.

$$\frac{3x}{5} - \frac{(x^2+5x+6)}{(10x+15)}$$

A. $\frac{(x^2+3)}{5(2x+3)}$

B. $\frac{5x^2+4x-6}{5(2x+3)}$

C. $\frac{-1(x^2+5x+3)}{25x(2x+3)}$

D. $5(x^3+5x^2-9)$

20. Combine the following.

$$\frac{x+4}{x}+\frac{(6x^2+9)}{3x}$$

A. $\frac{6x^2+x+13}{3x^2}$

B. $2x+13$

C. $\frac{2x^2+x+7}{x}$

D. $x+15$

21. Multiply.

$$\frac{4x^4}{3y^7}\times\frac{x^4y^7}{60xy}$$

A. $\frac{x^7}{45y^{-2}}$

B. $\frac{4x^{16}y^6}{180xy^8}$

C. $\frac{x^7y^2}{45}$

D. $\frac{x^7}{45y}$

22. Divide the polynomial and simplify.

$$(3x+4)\div\frac{(9x^2-16)}{7}$$

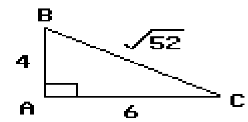
A. $\frac{7}{(3x-4)}$

B. $\frac{21x+28}{9x^2-16}$

C. $\frac{(3x+4)(9x^2-16)}{7}$

D. $x(9x^2-16)$

23. Triangle ABC is a right triangle. Which statement applies to this triangle?



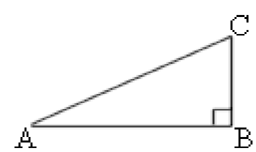
A. $\text{Tan B} = \frac{2}{\sqrt{5}}$

B. $\text{Cos B} = \frac{2\sqrt{13}}{13}$

C. $\text{Cos C} = \frac{3}{2}$

D. $\text{Sin C} = \frac{2}{3}$

24. Triangle ABC is a right triangle. Which statement applies to the triangle?



A. $\sin A = \frac{\cos A}{\tan A}$

B. $\cos A = \frac{\tan A}{\sin A}$

C. $\tan A = \frac{\sin A}{\cos A}$

D. $\tan A = \frac{\cos A}{\sin A}$

25. Evaluate. Round your answer to the nearest ten thousandth.

$\sin 300^\circ - \sin 173^\circ$

A. - 0.9879
B. 0.7986
C. 0.9879
D. - 0.7441

26. Evaluate. Round your answer to the nearest ten thousandth.

$\sin 12^\circ + \cos 25^\circ$

A. 1.4007
B. 1.1142
C. 1.8844
D. 0.6305

11th All Strands
Answer Key
03/12/2008

1.	**D**	Matrices
2.	**B**	Matrices
3.	**A**	Linear Programming
4.	**B**	Linear Programming
5.	**D**	Conic Sections
6.	**B**	Conic Sections
7.	**B**	Complex Numbers
8.	**C**	Complex Numbers
9.	**D**	Three Dimensional Space
10.	**B**	Three Dimensional Space
11.	**B**	Radians
12.	**C**	Radians
13.	**A**	Angles
14.	**D**	Angles
15.	**B**	Trigonometric Identities
16.	**B**	Trigonometric Identities
17.	**B**	Graphing Functions
18.	**B**	Graphing Functions
19.	**B**	Rational Expressions: Add/Subtract
20.	**C**	Rational Expressions: Add/Subtract
21.	**D**	Rational Functions: Multiply/Divide
22.	**A**	Rational Functions: Multiply/Divide
23.	**B**	Trigonometric Ratios - B
24.	**C**	Trigonometric Ratios - B
25.	**A**	Trigonometric Tables
26.	**B**	Trigonometric Tables

Study Guide

11th All Strands
03/12/2008

Matrices
A matrix is a rectangular arrangement of objects. Each object in a matrix is called an element. The plural form of matrix is matrices. One use of matrices is to store data, a second use is to describe transformations of various geometric figures, and a third use is to describe the pictures you see on television.

The figure below is an example of data stored in a matrix.

$$\begin{array}{r c l}
 & \begin{matrix}\text{Shirts} & \text{Pants} & \text{Shorts}\end{matrix} & \\
\begin{matrix}\text{Small}\\ \text{Medium}\\ \text{Large}\\ \text{Ex-Large}\end{matrix} & \left[\begin{matrix}10 & 8 & 9\\ 15 & 12 & 14\\ 13 & 18 & 13\\ 3 & 4 & \boxed{5}\end{matrix}\right] & \begin{matrix}\text{Row 1}\\ \text{Row 2}\\ \text{Row 3}\\ \text{Row 4}\end{matrix}\\
 & \begin{matrix}\text{Column 1} & \text{Column 2} & \text{Column 3}\end{matrix} &
\end{array}$$

$\boxed{5}$ is the element in the 4^{th} row and 3^{rd} column

The elements in a matrix are enclosed by large square brackets. This matrix has 4 rows and 3 columns. The matrix is said to have dimensions 4 by 3, written 4 x 3. A matrix with m rows and n columns has dimensions **m x n**. The dimensions of a matrix can also be called the order of the matrix. A square matrix is a matrix in which the number of rows equals the number of columns. Its dimensions are usually denoted as "n by n" or n x n.

Matrix Addition and Subtraction:

In order to perform matrix addition, the two matrices A and B must have the same dimensions (order). Once it has been determined that the matrices have the same dimensions, their sum A + B is the matrix in which each element is the sum of the corresponding elements in A and B.

Example 1: Add the matrices.

$$\overset{A}{\begin{bmatrix}2 & 4 & 3 & 1\\ 7 & 8 & 4 & 9\\ 6 & 3 & 2 & 8\end{bmatrix}} + \overset{B}{\begin{bmatrix}6 & 3 & 8 & 10\\ 4 & 4 & 1 & 3\\ 9 & 2 & 7 & 11\end{bmatrix}}$$

$$\overset{(1)}{\begin{bmatrix} \boxed{2+6} & 4+3 & 3+8 & 1+10 \\ 7+4 & 8+4 & 4+1 & 9+3 \\ 6+9 & 3+2 & 2+7 & 8+11 \end{bmatrix}} = A+B$$

$$\overset{(2)}{\begin{bmatrix} 8 & 7 & 11 & 11 \\ 11 & 12 & 5 & 12 \\ 15 & 5 & 9 & 19 \end{bmatrix}} = A+B$$

For the element in the 1^{st} row and 1^{st} column of matrix 1, $\boxed{2+6}$, 2 is the element in the 1^{st} row and 1^{st} column of matrix A while 6 is the element in the 1^{st} row and 1^{st} column of matrix B.

Step 1: Determine the order of each matrix. Matrix A has order of 3 x 4. Matrix B has order of 3 x 4. Since both matrices have the same order, they can be added together. To add the matrices together, we create a new matrix. Each element in the new matrix is the sum of the corresponding elements from A and B. If the element in the new matrix is in the 1st row and 1st column, then it is the sum of the elements in matrices A and B that are in the 1st row and 1st column.

Step 2: Determine each sum to complete the new matrix.

Matrix subtraction is similar to matrix addition. When you are given two matrices A and B, their difference A - B is the matrix in which each element is the difference of the corresponding elements in A and B.

Example 2: Subtract the matrices.

$$\overset{A}{\begin{bmatrix} 2 & 4 & 6 \\ 8 & -7 & 1 \\ -1 & 3 & -2 \end{bmatrix}} - \overset{B}{\begin{bmatrix} 3 & 1 & -2 \\ -4 & -8 & 3 \\ 2 & 1 & 6 \end{bmatrix}}$$

(1)

$$\begin{bmatrix} \boxed{2-3} & 4-1 & 6--2 \\ 8--4 & -7--8 & 1-3 \\ -1-2 & 3-1 & -2-6 \end{bmatrix} = A - B$$

(2)

$$\begin{bmatrix} -1 & 3 & 8 \\ 12 & 1 & -2 \\ -3 & 2 & -8 \end{bmatrix} = A - B$$

For the element in the 1^{st} row and 1^{st} column of matrix 1, $\boxed{2-3}$, 2 is the element in the 1^{st} row and 1^{st} column of matrix A while 3 is the element in the 1^{st} row and 1^{st} column of matrix B.

Step 1: Determine the order of each matrix. Matrix A has order of 3 x 3. Matrix B has order of 3 x 3. Since both matrices have the same order, they can be subtracted. To subtract the matrices, we create a new matrix. Each element in the new matrix is the difference of the corresponding elements from A and B. If the element in the new matrix is in the 1st row and 1st column, then it is the difference of the elements in matrices A and B that are in the 1st row and 1st column.

Step 2: Determine each difference to complete the new matrix.

HINT: When adding and subtracting matrices, the final matrix should have the same order (dimensions) as the two original matrices.

Multiplying a Matrix by a Scalar:

A scalar is a real number by which a matrix is multiplied. Scalar multiplication is the product of a scalar k and a matrix A. This product is a matrix kA in which each element is k times the corresponding element in A. Basically, this means that every number in the matrix is multiplied by the scalar.

Example 3: Multiply the matrix by the scalar.

(A)

$$2\begin{bmatrix} 2 & 4 & 7 & -1 \\ -3 & 6 & -5 & 8 \end{bmatrix}$$

(1)

$$\begin{bmatrix} \boxed{2\cdot 2} & 2\cdot 4 & 2\cdot 7 & 2\cdot -1 \\ 2\cdot -3 & 2\cdot 6 & 2\cdot -5 & 2\cdot 8 \end{bmatrix} = kA$$

(2)

$$\begin{bmatrix} 4 & 8 & 14 & -2 \\ -6 & 12 & -10 & 16 \end{bmatrix} = kA$$

For the element in the 1^{st} row and 1^{st} column of matrix 1, $\boxed{2\times 2}$, the first 2 is the scalar while the second 2 is the element in the 1^{st} row and 1^{st} column of matrix A.

Step 1: Multiply each element in the matrix by the scalar (k = 2).

Step 2: Determine each product to complete the new matrix.

Matrix Multiplication:

Matrix multiplication is the process of

multiplying one matrix by another matrix. Matrix multiplication is completed by multiplying a row by a column. Multiply the first element in the row by the first element in the column, the second element in the row by the second element in the column, and so on. Finally, find the sum of the resulting products. Two matrices can only be multiplied if the number of columns of A equals the number of rows of B. See the diagram below.

$$\begin{bmatrix}3&4\\-5&6\end{bmatrix}\begin{bmatrix}6&-7&8\\9&-8&4\end{bmatrix} \qquad \begin{bmatrix}6&-7&8\\9&-8&4\end{bmatrix}\begin{bmatrix}3&4\\-5&6\end{bmatrix}$$

$$2\times\underbrace{2\qquad 2}_{\text{Equal}}\times 3 \qquad\qquad 2\times\underbrace{3\qquad 2}_{\text{Not Equal}}\times 2$$

These matrices can be multiplied. These matrices cannot be multiplied.

One important concept in matrix multiplication is the ability to determine the order (dimensions) of matrix AB. The following is an explanation of how to determine the order of matrix AB.

Definition:
Suppose A is an m x n matrix and B is an n x p matrix. Then the order (dimensions) of the matrix AB equals m x p.

Example 4: Matrix A has order 5 x 3. Matrix B has order 3 x 2. Find the order of the resulting matrix AB.

(1)
(2)
5 x 3 3 x 2
5 x 2
m x n n x p
m x p

Step 1: Write the dimensions of the two matrices in the order they are to be multiplied. Remember, matrix A has order m x n and matrix B has order n x p.
Step 2: If the n value is the same for both matrices, the product of AB exists. Since n = 3 for both matrices, the order of matrix AB is 5 x 2.

Example 5: Matrix A has order 4 x 5. Matrix B has order 3 x 4. Find the order of the resulting matrix AB.

(1)
(2)
4 x 5 3 x 4
Does not exist
m x n n x p

Step 1: Write the dimensions of the two matrices in the order they are to be multiplied. Remember, matrix A has order m x n and matrix B has order n x p.
Step 2: Since the value of n is not the same for both matrices, it is impossible to multiply A by B and matrix AB does not exist.

Now that we can determine the order of matrix AB, it will be easier to multiply A and B.

Example 6: Multiply the matrices.

$$\underset{(A)}{\begin{bmatrix}3&4\\-5&6\end{bmatrix}} \times \underset{(B)}{\begin{bmatrix}6&-7&8\\9&-8&4\end{bmatrix}}$$

(1)

$$AB = \begin{bmatrix} 3 & 4 \\ -5 & 6 \end{bmatrix} \times \begin{bmatrix} 6 & -7 & 8 \\ 9 & -8 & 4 \end{bmatrix}$$

$$2 \times 2 \qquad 2 \times 3$$

(2)

$$\begin{bmatrix} \boxed{3 \quad 4} \\ -5 \quad 6 \end{bmatrix} \times \begin{bmatrix} \boxed{\begin{matrix} 6 \\ 9 \end{matrix}} & \begin{matrix} -7 & 8 \\ -8 & 4 \end{matrix} \end{bmatrix} = \begin{bmatrix} \boxed{54} & - & - \\ - & - & - \end{bmatrix}$$

$$3 \cdot 6 + 4 \cdot 9 = 18 + 36 = 54$$

(3)

$$\begin{bmatrix} 3 \quad 4 \\ \boxed{-5 \quad 6} \end{bmatrix} \times \begin{bmatrix} \boxed{\begin{matrix} 6 \\ 9 \end{matrix}} & \begin{matrix} -7 & 8 \\ -8 & 4 \end{matrix} \end{bmatrix} = \begin{bmatrix} 54 & - & - \\ \boxed{24} & - & - \end{bmatrix}$$

$$-5 \cdot 6 + 6 \cdot 9 = -30 + 54 = 24$$

(4)

$$\begin{bmatrix} 3 \quad 4 \\ \boxed{-5 \quad 6} \end{bmatrix} \times \begin{bmatrix} \begin{matrix} 6 & -7 \\ 9 & -8 \end{matrix} & \boxed{\begin{matrix} 8 \\ 4 \end{matrix}} \end{bmatrix} = \begin{bmatrix} 54 & -53 & 40 \\ 24 & -13 & \boxed{-16} \end{bmatrix}$$

$$-5 \cdot 8 + 6 \cdot 4 = -40 + 24 = -16$$

(5)

$$AB = \begin{bmatrix} 54 & -53 & 40 \\ 24 & -13 & -16 \end{bmatrix}$$

Step 1: Find the dimensions (order) of AB. Since a 2 x 2 matrix is being multiplied by a 2 x 3 matrix, the order of the product is 2 x 3.
Step 2: The product of row 1 of A and column 1 of B is 3 x 6 + 4 x 9 = 18 + 36 = 54. This is put in the 1st row and 1st column of the answer.
Step 3: The product of row 2 of A and column 1 of B is -5 x 6 + 6 x 9 = -30 + 54 = 24. This is put in the 2nd row and 1st column of the answer.
Step 4: The element in the 2nd row, 3rd column of AB is found by multiplying the 2nd row of A by the 3rd column of B. -5 x 8 + 6 x 4 = -40 + 24 = -16
Step 5: The other three elements of AB are found using this row by column pattern.

Finding the Determinant of a Matrix:

The determinant of a matrix is a real number and can be used to determine whether or not a matrix has an inverse. The determinant can only be found for a square matrix. We abbreviate the word "determinant" as "det."

Definition: Determinant of a 2×2 matrix

Given a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ac$

Another notation for the determinant of matrix A follows.

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Example 7: Find the determinant of the matrix.

$$A = \begin{bmatrix} 4 & 6 \\ -2 & 3 \end{bmatrix}$$

(1) a = 4, b = 6, c = -2, d = 3
(2) determinant A = ad - bc
(3) det A = (4 x 3) - (6 x -2)
(4) det A = 12 - -12 = 12 + 12
= 24

Step 1: Using the definition of a determinant above, find the values of a, b, c, and d.
Step 2: Select the correct formula for finding the determinant.
Step 3: Substitute the values of a, b, c, and d into the formula for the determinant.
Step 4: Following the order of operations, multiply 4 by 3 to get 12. Then multiply 6 by -2 to get -12.
12 - -12 becomes 12 + 12. Add 12 and 12 to get 24.

The determinant of matrix A equals 24.

The determinant of a 3 x 3 matrix is as follows:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - d\begin{vmatrix} b & c \\ h & i \end{vmatrix} + g\begin{vmatrix} b & c \\ e & f \end{vmatrix}$$

Example 8: Find the determinant of the matrix.

$$\begin{bmatrix} 4 & -1 & 0 \\ 3 & 2 & -2 \\ -5 & -7 & 5 \end{bmatrix}$$

(1) $a = 4, b = -1, c = 0, d = 3, e = 2, f = -2, g = -$

(2) $4\begin{vmatrix} 2 & -2 \\ -7 & 5 \end{vmatrix} - 3\begin{vmatrix} -1 & 0 \\ -7 & 5 \end{vmatrix} + -5\begin{vmatrix} -1 & 0 \\ 2 & -2 \end{vmatrix}$

(3) $2[2(5) - -2(-7)] - 3[-1(5) - 0(-7)] + -5[-$

(4) $4[10 - 14] - 3[-5 + 0] + -5[2 - 0]$

(5) $4[-4] - 3[-5] + -5[2]$

(6) $-16 + 15 + -10 = -11$

Step 1: Using the definition of a 3 x 3 determinant, find the values of a, b, c, d, e, f, h, and i.
Step 2: Substitute the values of a, b, c, d, e, f, h, and i into the formula for the determinant of a 3 x 3 matrix.
Step 3: Using the definition of a 2 x 2 matrix, find the determinants of the 2 x 2 matrices.
Step 4: Following the order of operations, multiply the values inside the parentheses.
Step 5: Following the order of operations, add or subtract the values inside the parentheses.
Step 6: Multiply the values and then add them together to get -11.

Answer: -11

Solving a System of Equations Using Matrices:

A system of equations is a group of two or more equations that are related to each other. The solution set of a system of equations is often an ordered pair of the form (x, y). That ordered pair is a solution to both equations of the system. The ordered pair (-2, 12) is a solution of the system:

$$x + y = 10$$
$$2x + y = 8$$

because (-2, 12) satisfies both equations as shown below.

$x + y = 10$ and $2x + y = 8$

$-2 + 12 = 10$ and $2(-2) + 12 = 8$

$10 = 10$ $8 = 8$

true true

A solution set of a system of 3 equations is often an ordered triple of the form (x, y, z). If (3, 4, 5) is the solution to a system, it is acceptable to express the answer as x = 3, y = 4, and z = 5.

Solving a system of equations using matrices involves writing the equations in the form of a matrix and then solving for the value of the variables using row elimination.

Example 9: Solve the following system of equations using matrices.

$$2x + 4y + z = 5$$
$$x + 3y + 2z = 2$$
$$4x - y - z = -3$$

It is possible to represent this system as a matrix equation.

$$\begin{bmatrix} 2 & 4 & 1 \\ 1 & 3 & 2 \\ 4 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix}$$

coefficient matrix constant matri:

The coefficient matrix is the matrix which represents the coefficients of the variables in each equation. A coefficient is a number that is multiplied by a variable. The constant matrix is the matrix which represents the constant terms that the matrices equal.

There are two methods for solving a system of equations using matrices. One of those methods involves determining the inverse of the matrix, multiplying the matrix by the inverse, and solving for the values of the variables. The other way to solve systems of equations using matrices is using the process of row elimination. The method we are going to explain is row elimination. We chose this method because it is more straight-forward and involves less confusing steps.

Row elimination is the process of performing addition, subtraction, multiplication, and division on the rows of a matrix until the matrix is in the form of the identity matrix.

Identity 3×3 matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Identity 2×2 matri:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(1) \left[\begin{array}{ccc|c} 2 & 4 & 1 & 5 \\ 1 & 3 & 2 & 2 \\ 4 & -1 & -1 & -3 \end{array}\right]$$

Step 1: The first step in solving systems of equations using row elimination is to write

the coefficient matrix and the constant matrix as one matrix. This is accomplished by writing the coefficient matrix, drawing a vertical line after the last column of the coefficient matrix, and then writing the constant matrix. The vertical line separates the two matrices because they will need to be broken apart again once the coefficient matrix is in the form of the identity matrix. The vertical line is sometimes drawn as a dotted line instead of a solid one.

(2) Row 1 − Row 2

$$\left[\begin{array}{ccc|c} 2-1 & 4-3 & 1-2 & 5-2 \\ 1 & 3 & 2 & 2 \\ 4 & -1 & -1 & -3 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 1 & 3 & 2 & 2 \\ 4 & -1 & -1 & -3 \end{array}\right]$$

Step 2: Subtract Row 2 from Row 1 to make the element in the 1st row and 1st column equal 1. When adding or subtracting a row from another row, add or subtract the corresponding elements. For future reference, the new row 1 that is created by this operation is now referred to as "row 1."

(3) Row 2 − Row 1

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 1-1 & 3-1 & 2--1 & 2-3 \\ 4 & -1 & -1 & -3 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 2 & 3 & -1 \\ 4 & -1 & -1 & -3 \end{array}\right]$$

Step 3: Subtract Row 1 from Row 2 to make the element in the 2nd row and 1st column equal 0. If you look at the new matrix, you will notice that part of it is now in the form of the identity matrix. We must be careful not to change the two elements we have already put in the form of the identity matrix.

(4) Row 3 − 4 (Row 1)

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 2 & 3 & -1 \\ 4-4 & -1-4 & -1--4 & -3-12 \end{array}\right] = \left[\begin{array}{cc} 1 & 1 \\ 0 & 2 \\ 0 & -5 \end{array}\right.$$

Step 4: Multiply Row 1 by 4 and subtract that product from Row 3. This will make the element in the 3rd row and 1st column equal to 0. Now the 1st column is in the form of the identity matrix. When multiplying a row by a number, remember to multiply each element in the row by the stated number.

(5) Row 3 + 2 (Row 2)

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 2 & 3 & -1 \\ 0+0 & -5+4 & 3+6 & -15+-2 \end{array}\right] = \left[\begin{array}{cc} 1 & 1 \\ 0 & 2 \\ 0 & -1 \end{array}\right.$$

Step 5: Multiply Row 2 by 2 and add that product to Row 3. This step does not put any element of the matrix in the form of the identity matrix, but it does allow us to add Row 3 to Row 1 and Row 3 to Row 2 which

will put elements in both of those rows in the form of the identity matrix.

(6) Row 1 + Row 3 and Row 2 + Row 3

$$\left[\begin{array}{ccc|c} 1+0 & 1+{}^{-}1 & {}^{-}1+9 & 3+{}^{-}17 \\ 0+0 & 2+{}^{-}1 & 3+9 & {}^{-}1+{}^{-}17 \\ 0 & {}^{-}1 & 9 & {}^{-}17 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 8 & {}^{-}14 \\ 0 & 1 & 12 & {}^{-}18 \\ 0 & {}^{-}1 & 9 & {}^{-}17 \end{array}\right]$$

<u>Step 6</u>: Add Row 3 to Row 1 to make the element in the 1st row and 2nd column equal 0. Add Row 3 to Row 2 to make the element in the 2nd row and 2nd column equal 1. Half of the matrix is now in the form of the identity matrix.

(7) Row 3 + Row 2

$$\left[\begin{array}{ccc|c} 1 & 0 & 8 & {}^{-}14 \\ 0 & 1 & 12 & {}^{-}18 \\ 0+0 & {}^{-}1+1 & 9+12 & {}^{-}17+{}^{-}18 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 8 & {}^{-}14 \\ 0 & 1 & 12 & {}^{-}18 \\ 0 & 0 & 21 & {}^{-}35 \end{array}\right]$$

<u>Step 7</u>: Add Row 2 to Row 3 to make the element in the 3rd row and 2nd column equal 0.

(8) Row 3 ÷ 7

$$\left[\begin{array}{ccc|c} 1 & 0 & 8 & {}^{-}14 \\ 0 & 1 & 12 & {}^{-}18 \\ 0\div 7 & 0\div 7 & 21\div 7 & {}^{-}35\div 7 \end{array}\right] = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right.$$

<u>Step 8</u>: Divide Row 3 by 7 to reduce the size of the numbers. Dividing by 7 was chosen because 21 and -35 are both divisible by 7.

(9) Row 1 − 3(Row 3) and Row 2 − 4(Row3

$$\left[\begin{array}{ccc|c} 1-0 & 0-0 & 8-9 & {}^{-}14-{}^{-}15 \\ 0-0 & 1-0 & 12-12 & {}^{-}18-{}^{-}20 \\ 0 & 0 & 3 & {}^{-}5 \end{array}\right] = \left[\right.$$

<u>Step 9</u>: Multiply Row 3 by 3 and subtract that product from Row 1. This will make the element in the 1st row and 3rd column equal -1 (while this is not matrix identity form, it will make the element easier to manipulate later). Then multiply Row 3 by 4 and subtract that product from Row 2. This will make the element in the 2nd row and 3rd column equal 0. Now the matrix is almost in the form of the identity matrix.

(10) Row 3 ÷ 3

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 \\ 0\div 3 & 0\div 3 & 3\div 3 & -5\div 3 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & \frac{-5}{3} \end{array}\right]$$

Step 10: Divide Row 3 by 3. This will make the element in the 3rd row and 3rd column equal 1. Now, only one element of the coefficient matrix is not in matrix identity form.

(11) Row 1 + Row 3

$$\left[\begin{array}{ccc|c} 1+0 & 0+0 & {}^{-}1+1 & 1+\frac{-5}{3} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & \frac{-5}{3} \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{-2}{3} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & \frac{-5}{3} \end{array}\right]$$

Step 11: Add Row 3 to Row 1 to make the element in the 1st row and 3rd column equal 0. Now the coefficient matrix is in the form of the 3 x 3 identity matrix.

$$(12)\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{-2}{3} \\ 2 \\ \frac{-5}{3} \end{bmatrix}$$

Step 12: Separate the coefficient (identity) matrix from the constant matrix. Write the matrices as a matrix equation once again.

$$(13)\quad \begin{aligned} x &= -\tfrac{2}{3} \\ y &= 2 \\ z &= -\tfrac{5}{3} \end{aligned}$$

Step 13: Multiply the coefficient matrix and the variable matrix. The final result is the solution to the system of equations.

It is important to note that row elimination has no distinct pattern. It is accomplished simply by manipulating the rows of the matrix to try to get the identity matrix.

Linear Programming

Linear programming is the process of finding maximum or minimum values under limiting conditions or constraints. Linear programming is used to find the best possible combination of two or more variable quantities that determine the value of another quantity. The constraints are the conditions that limit the possible maximum and minimum values. Linear programming is often used to determine the amount of a product a company should manufacture/use to minimize/maximize profits.

One of the most important concepts needed to do linear programming is graphing inequalities. The following is a detailed description of how to graph an inequality.

Example 1: Graph the following inequality.

$$x + 3y \le 6$$

Step 1: To graph an inequality, we must pretend the inequality is an equation. We would pretend that $x + 3y \le 6$ was $x + 3y = 6$. Step 2: Now, we

graph $x + 3y = 6$. To do this, we create a table of values. A table of values is a table that lists the values of x and corresponding values of y. To determine the values of x that should be represented in the table, we simply choose 3 values. It is helpful if one of the values of x is positive, one is negative, and the third value of x can be positive, negative, or zero.

table of values

X	Y

choose any 3 values for X

X	Y
-3	
0	
3	

To determine the values of y, substitute each value of x into $x + 3y = 6$ and solve for y.

$$x = -3 \qquad -3 + 3y = 6 \qquad \underline{+3 \qquad +3} \qquad \frac{3y}{3} = \frac{9}{3} \qquad y = 3$$

$$x = 0 \qquad 0 + 3y = 6 \qquad \frac{3y}{3} = \frac{6}{3} \qquad y = 2$$

$$x = 3 \qquad 3 + 3y = 6 \qquad \underline{-3 \qquad -3} \qquad \frac{3y}{3} = \frac{3}{3} \qquad y = 1$$

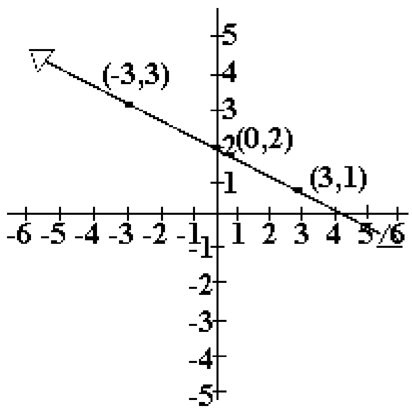
X	Y
-3	3
0	2
3	1

This is the final table of values.

Step 3: Now that we have a table of values, we can make ordered pairs (coordinate points). All ordered pairs are written in the form (x, y). The table of values can be easily written as ordered pairs:

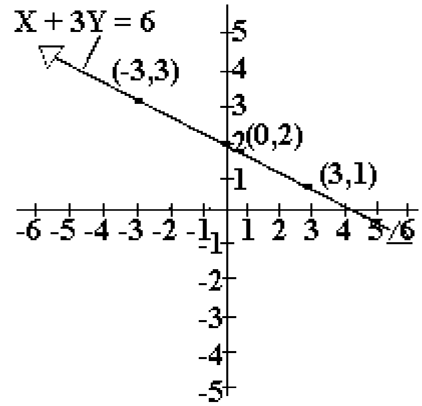
X	Y		Ordered Pair
-3	3	→	$(-3, 3)$
0	2	→	$(0, 2)$
3	1	→	$(3, 1)$

Step 4: Now, we graph the ordered pairs on a coordinate plane. Remember, the x-value of the ordered pair determines the left (-x) or right (+x) movement, while the y-value determines the up (+y) or down (-y) movement. Always graph the x-value first, then move from that point up or down to the y-value.



Step 5: After the three points are graphed, connect them with a straight line. It is often helpful to label this line with its equation. (See the graph above.)

Step 6: The graph of an inequality is always a shaded half-plane. To determine the half of the coordinate plane that needs to be shaded, choose a test point that does not lie on the graph of the line (usually the test point is (0,0)). Substitute the test point into the original inequality and simplify. If the answer is true, shade from the graph of the line toward the test point. If the answer is false, shade the graph of the line away from the test point.



NOTE: It may be helpful to note that the inequality $x + 3y < 6$ would require a dotted boundary line instead of a solid boundary line because the value of $x + 3y$ can never be equal to 6.

Now that you know how to graph an inequality, we can find the maximum or minimum of a function.

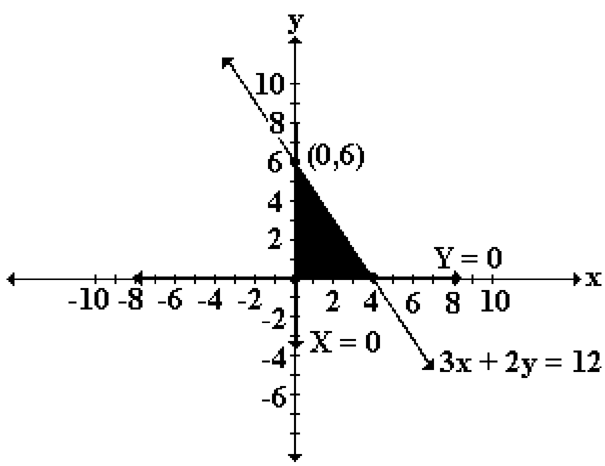
Example 2: Find the minimum and maximum values of $C = 3x + 4y$ using the following constraints.

$$x \geq 0$$
$$y \geq 0$$
$$3x + 2y \leq 12$$

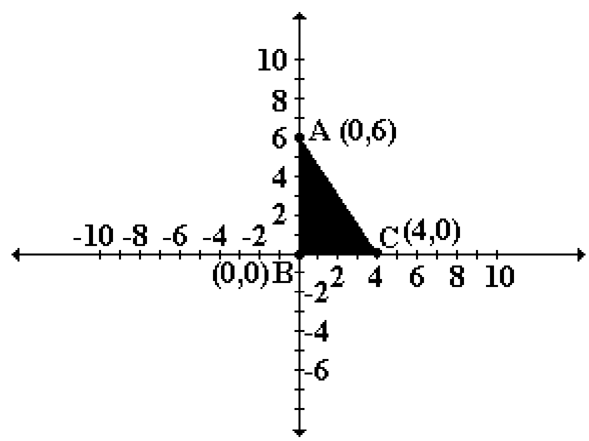
Step 1: The equation $C = 3x + 4y$ is considered the cost function. This is the function we are trying to minimize and maximize. We do not need to use this function until we determine possible values for the variables x and y.

Step 2: To determine possible values for x and y, we must graph all the constraints on one coordinate plane.

Step 3: Shade in the feasible region. The feasible region is the set of solutions to a system of linear inequalities. The Linear-Programming Theorem states that if the feasible region of a linear programming problem is convex, then the maximum or minimum quantity is determined at one of the vertices of the region. The feasible region of the constraints graphed above is shaded in dark. The feasible region is the region bounded by the graphs such that the shading from all graphs cross each other.



Step 4: Determine the vertices of the feasible region. A vertex of a feasible region is a corner point where two of the boundary lines graphed intersect each other. In the graph below, A, B, and C are the vertices of the feasible region.



Vertex A is determined by finding the point where x = 0 and 3x + 2y = 12 intersect each other. Calculate the value of y in 3x + 2y = 12 when x = 0.

$$3(0) + 2y = 12$$
$$2y = 12$$
$$y = 6$$

Vertex A is the coordinate point (0, 6).

Vertex B is determined by finding the point where x = 0 and y = 0 intersect each other. Vertex B is the coordinate point (0,0), the

origin.

Vertex C is determined by finding the point where $y = 0$ and $3x + 2y = 12$ intersect each other. Calculate the value of x in $3x + 2y = 12$ when $y = 0$.

$$3x + 2(0) = 12$$
$$3x = 12$$
$$x = 4$$

Vertex C is the coordinate point (4,0).

Step 5: Determine the minimum and maximum values of the equation $C = 3x + 4y$. The Linear-Programming Theorem says that the minimum and maximum values of $C = 3x + 4y$ occur at vertices of the feasible region. Evaluate the equation $C = 3x + 4y$ at each vertex. This involves substituting the values of x and y at each vertex into $C = 3x + 4y$ and calculating the value of C.

Vertex A (0,6): $3(0) + 4(6) = 0 + 24 = 24$
Vertex B (0,0): $3(0) + 4(0) = 0 + 0 = 0$
Vertex C (4,0): $3(4) + 4(0) = 12 + 0 = 12$

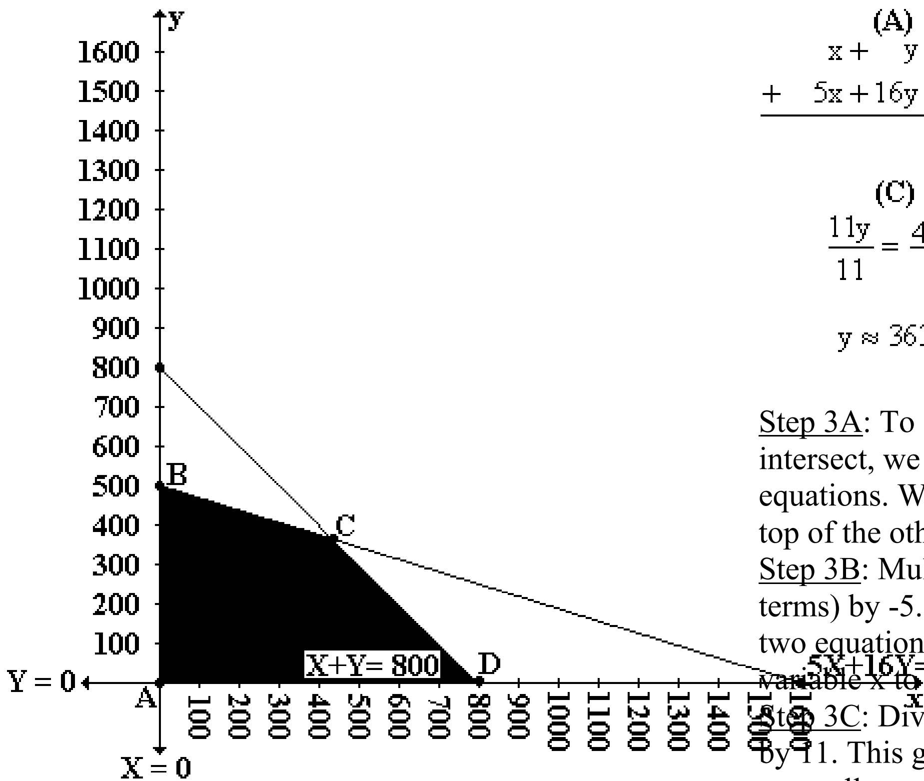
The maximum value of $C = 3x + 4y$ is 24 because that is the largest number calculated when the vertices are substituted into the equation. The minimum value of $C = 3x + 4y$ is 0 because that is the smallest number calculated when the vertices are substituted into the equation.

Example 3: A clothing factory produces casual shirts and dress shirts. Not more than 800 shirts can be made per hour and production may not cost over \$8,000 per hour. It costs \$5 to make a casual shirt and \$16 per dress shirt. The profit is \$1.25 per casual shirt and \$4.00 per dress shirt. If x is the number of casual shirts made per hour and y is the number of dress shirts made per hour, what is the maximum value of the profit function $P = 1.25x + 4y$?

Step 1: Determine the constraints of the problem. Since the problem is about shirts and it is not possible to have a negative number of shirts, $x \geq 0$ and $y \geq 0$. We also know that 800 is the maximum number of shirts that can be produced per hour, so $x + y \leq 800$. The final constraint is that production cost may not exceed \$8,000, so $5x + 16y \leq 8000$.

$$x \geq 0$$
$$y \geq 0$$
$$x + y \leq 800$$
$$5x + 16y \leq 8000$$

Step 2: Follow the steps in Examples 1 and 2 to graph the four constraints. Then shade in the feasible region. Since x and y must both be greater than or equal to 0, it is only necessary to show the part of the coordinate plane where x and y are positive.



Step 3: Determine the vertices of the feasible region. Three of the four vertices are easily distinguishable. Vertex A is (0,0), Vertex B is (0,500), and Vertex D is (800,0). To determine the coordinate point of Vertex C, we need to determine where the graphs of x + y = 800 and 5x + 16y = 8000 intersect.

(A)

$$\begin{array}{r} x + \quad y = \ 800 \\ + \quad 5x + 16y = 8000 \\ \hline \end{array}$$

(B)

$$\begin{array}{r} -5x + -5y = -400 \\ + \quad 5x + 16y = \ 800 \\ \hline 11y = \ 400 \end{array}$$

(C)

$$\frac{11y}{11} = \frac{4000}{11}$$

$$y \approx 363.64$$

(D)

$$\begin{array}{r} x + y = \ 800 \\ x + 363.64 = \ 800 \\ -363.64 \quad -363.6 \\ \hline x = \ 436.3 \end{array}$$

Step 3A: To determine where the two graphs intersect, we must solve the system of equations. Write the two equations one on top of the other.
Step 3B: Multiply the top equation (all three terms) by -5. This will allow us to add the two equations together and eliminate the variable x to get 11y = 4000.
Step 3C: Divide both sides of the equation by 11. This gives us the value of y. (It is generally respected to round decimal numbers to two decimal places.)
Step 3D: Substitute the value of y into one of the two equations. The equation x + y = 800 was chosen because it is a simpler equation. After substituting the value of y into the equation, solve the equation for the value of x by subtracting 363.64 from each side of the equation to get x = 436.36 and y = 363.64.

The four vertices of the feasible region are:

Vertex A: (0,0)
Vertex B: (0,500)
Vertex C: (436.36,363.64)
Vertex D: (800,0)

Step 4: Substitute the x and y values from each vertex into the profit function P = 1.25x + 4y to determine the maximum value of the function.

Vertex A (0,0): 1.25(0) + 4(0) = 0 + 0 = 0
Vertex B (0,500): 1.25(0) + 4(500) = 0 + 2000 = 2000
Vertex C (436.36,363.64): 1.25(436.36) + 4(363.64) = 545.45 + 1454.56 = 2000.01
Vertex D (800,0): 1.25(800) +

$4(0) = 1000 + 0 = 1000$

The **maximum value** of the profit function is **2000.01** because that is the largest result of substituting the vertices into the profit equation P = 1.25x + 4y.

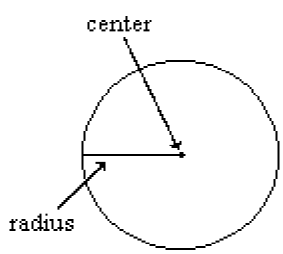
NOTE: The maximum value may be larger than 2000.01 if 4000/11 is not rounded in Step 3C.

Conic Sections

A conic section is a cross-section of a double cone. Conic sections can be circles, hyperbolas, parabolas, or ellipses.

Writing the Equation of A Circle:

A circle is a set of points in a plane that lie the same distance from a fixed point called the center. The radius of a circle is the distance from the center of the circle to any point on the circle.



The standard form of the equation of a circle with center at (h, k) is $(x-h)^2+(y-k)^2=r^2$, where r is the radius.

Example 1: Write the equation of a circle with center at (4, 2) and radius equal to 3.

(1) $(x-4)^2+(y-2)^2=(3)^2$

(2) $(x-4)^2+(y-2)^2=9$

Step 1: Substitute the values into the equation (h = 4, k = 2, r = 3).
Step 2: Square the value of the radius.

Example 2: Write the equation of a circle with center at (-1, -8) and radius equal to 1.

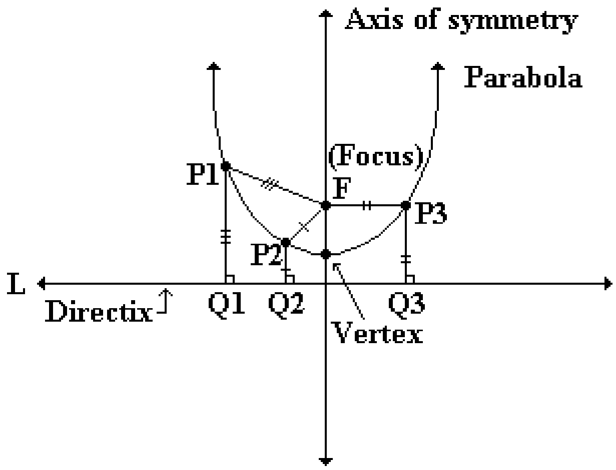
(1) $(x--1)^2+(y--8)^2=(1)^2$

(2) $(x+1)^2+(y+8)^2=1$

Step 1: Substitute the values into the equation (h = -1, k = -8, r = 1).
Step 2: Square the value of the radius and simplify the values in the parentheses.

Writing the Equation of a Parabola:

A parabola is the graph of a symmetrical curve. Every parabola has a focus. The focus is the point (not on the parabola) from which every point on the parabola is equidistant. Parabolas also have a directix. The directix is the line whose distance to any point on a parabola is equal to the distance from that point to the focus. In the diagram below, the distance from point P1 to the focus is the same as the distance from point P1 to the directix.



The standard form of a parabola with vertex at (h, k) is:

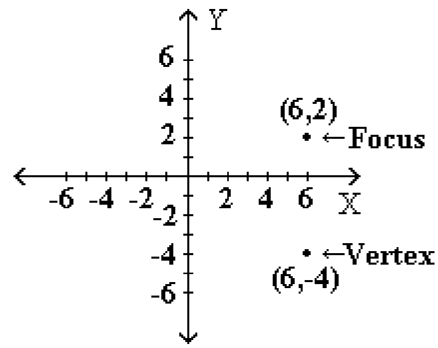
Vertical axis : $(x-h)^2 = 4p(y-k)$; When the focus $= (h, k+p)$

Horizontal axis : $(y-k)^2 = 4p(x-h)$; When the focus $= (h+p, k)$

In the equation for the standard form of a parabola, p is the distance from the vertex to the focus. If the parabola opens downward or to the left, p will be negative.

Example 3: Write the equation of the parabola in standard form with a vertex at (6, -4) and focus at
(6, 2).

Step 1: Determine the axis of the parabola. To do this, sketch the vertex and the focus of the parabola.



The axis of this parabola is vertical. We can determine this by noting that if we connect the vertex and focus of the parabola we will get a vertical line.

Step 2: Determine the value of p by determining the distance between the vertex and the focus. The distance between 2 and -4 is 6, so p = 6.

Step 3: Determine the values of h and k. h = 6 and k = -4 (these values are found using the vertex).

(4) $(x-6)^2 = 4(6)(y--4)$

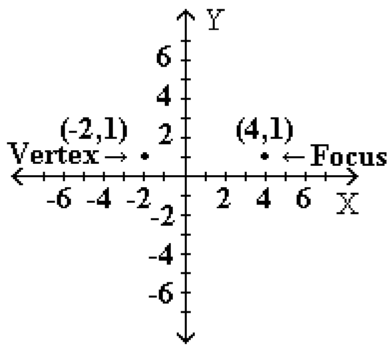
(5) $(x-6)^2 = 24(y+4)$

Step 4: Substitute h = 6, k = -4, and p = 6 into the equation for the standard form of a parabola with a vertical axis.

Step 5: Multiply 6 by four and simplify the values in the parentheses.

Example 4: Write the equation of the parabola with a vertex at (-2, 1) and focus at (4,1).

Step 1: Determine the axis of the parabola. To do this, sketch the vertex and the focus of the parabola.



The axis of this parabola is horizontal. We can determine this by noting that if we connect the vertex and focus of the parabola we will get a horizontal line.

Step 2: Determine the value of p by determining the distance between the vertex and the focus. The distance between -2 and 4 is 6, so p = 6.

Step 3: Determine the values of h and k. h = -2 and k = 1 (these values are found by looking at the vertex).

$$(4)\ \ (y-1)^2 = 4(6)(x - -2)$$

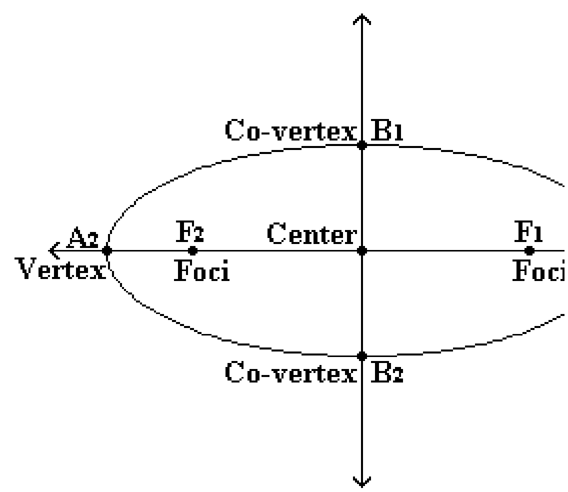
$$(5)\ \ (y-1)^2 = 24(x+2)$$

Step 4: Substitute h = -2, k = 1, and p = 6 into the equation for the standard form of a parabola with a horizontal axis.

Step 5: Multiply 6 by four and simplify the values in the parentheses.

Writing the Equation of an Ellipse:

An ellipse is a closed two-dimensional plane figure that is oval in shape. Every ellipse has two axes. The two axes lie on the symmetry lines and intersect at the center of the ellipse. One of the axes is the major axis. The major axis contains the foci, has two vertices of the ellipse as its endpoints, and is always longer. The other axis of an ellipse is called the minor axis. The minor axis does not contain the foci and has two vertices of the ellipse (the co-vertices) as its endpoints. (See the diagram below.)



The standard form of the equation of an ellipse:

$$\text{Vertical axis: } \frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

$$\text{Horizontal axis: } \frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

• (h, k) is the center of the ellipse, a > b, and $c^2 = a^2 - b^2$. • To find the center of the ellipse, find the midpoint of the vertices.

• The distance from the vertices to the center is $\pm a$. • The distance from the center to the co-vertices is $\pm b$. • The distance from the foci to the center is $\pm c$.

Use the following formula to find the midpoint of two points, (x_1, y_1)

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Example 5: Write the equation of the ellipse with vertices at (0, 5) and (0, -5) and foci at (0, 3) and
(0, -3).

Step 1: Determine the center of the ellipse by finding the midpoint of the vertices using the midpoint formula given above.

$$\text{(A) } \left(\frac{0+0}{2}, \frac{5+-5}{2}\right) \qquad \text{(B) } \left(\frac{0}{2}, \frac{0}{2}\right) \qquad \text{(C) } (0, 0)$$

Step 1A: Substitute the values from the vertices into the formula for the midpoint of a line.
Step 1B: Simplify the numerators of the fractions (0 + 0 = 0 and 5 + -5 = 0).
Step 1C: The midpoint of the vertices is (0, 0).

The center is (0, 0) and h = 0, k = 0.

Step 2: Determine whether the ellipse has a vertical or horizontal major axis. The foci are on the vertical axis of the ellipse, so the appropriate equation is the standard form of an ellipse with a vertical axis.

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1 \text{, where } c^2 = a^2 - b^2$$

Step 3: Determine the values of a, b, and c. a is the distance from the vertex to the center, b is the distance from the co-vertex to the center, and c is the distance from the foci to the center. In this case, we can determine a = 5 and c = 3. To determine the value of b, we substitute a = 5 and c = 3 into the equation $c^2 = a^2 - b^2$.

(A)
$$c^2 = a^2 - b^2$$
$$(3)^2 = (5)^2 - b^2$$

(B)
$$9 = 25 - b^2$$
$$16 = b^2$$

(C)
$$\sqrt{16} = \sqrt{b}$$
$$\pm 4 = b$$
$$4 = b$$

Step 3A: Substitute the values a = 5 and c = 3 into the equation. Then square each value.
Step 3B: Solve for b^2. Step 3C: Take the square root of each side of the equation. Only the positive value of the square root is needed, so b = 4.

Step 4: Substitute the values of h = 0, k = 0, a = 5, and b = 4 into the equation for the standard form of an ellipse with a vertical axis and simplify to determine the equation of this ellipse.

(A)

$$\frac{(x-0)^2}{(4)^2}+\frac{(y-0)^2}{(5)^2}=1$$

(B)

$$\frac{x^2}{16}+\frac{y^2}{25}=1$$

Example 6: Write the equation of the ellipse with vertices at (5, 0) and (-3, 0) and co-vertices at
(1, 2) and (1, -2).

Step 1: Determine the center of the ellipse. Either the vertices or the co-vertices can be used to determine the center. In this example, the vertices will be used. Use the formula for the midpoint to determine the center.

(A)

$$\left(\frac{5+-3}{2},\frac{0+0}{2}\right)$$

(B)

$$\left(\frac{2}{2},\frac{0}{2}\right)$$

(C)

$$(1,0)$$

Step 1A: Substitute the values from the vertices into the formula for the midpoint of a line.
Step 1B: Simplify the numerators of the fractions (5 + -3 = 2 and 0 + 0 = 0).
Step 1C: The midpoint of the vertices is (1, 0).

The center is (1, 0) and h = 1, k = 0.

Step 2: Determine whether the ellipse has a vertical or horizontal major axis. The vertices are on the horizontal axis of the ellipse, so the appropriate equation is the standard form of an ellipse with a horizontal axis.

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1\text{, where } c^2=a^2-b^2$$

Step 3: Determine the values of a and b. a is the distance from the vertex to the center and b is the distance from the co-vertex to the center. In this case, we can determine a = 4 and b = 2. We do not need to determine the value of c this time, because we know the locations of the vertices and the co-vertices.

Step 4: Substitute the values of h = 1, k = 0, a = 4, and b = 2 into the equation for the standard form of an ellipse with a horizontal axis and simplify to determine the equation of this ellipse.

(A)

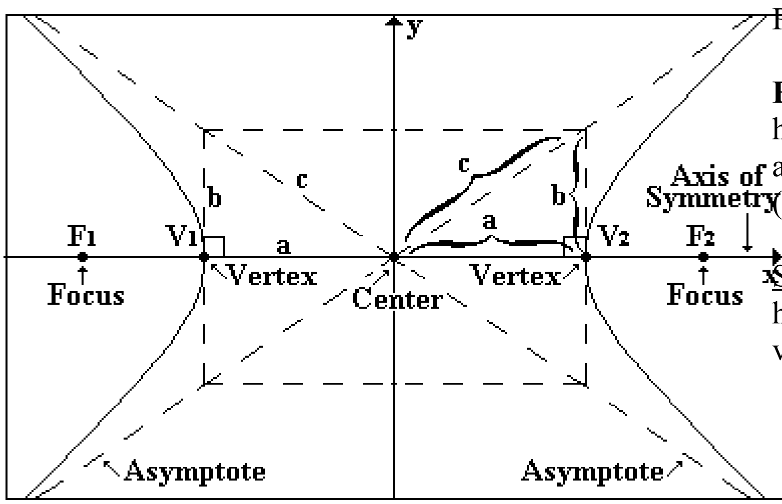
$$\frac{(x-1)^2}{(4)^2}+\frac{(y-0)^2}{(2)^2}=1$$

(B)

$$\frac{(x-1)^2}{16}+\frac{y^2}{4}=1$$

Writing the Equation of a Hyperbola:

A hyperbola is a plane figure that has two branches and is composed of the set of all points in which the difference of their distances from the two fixed points is a constant. Hyperbolas also have vertices, foci, an axis of symmetry, and a center. The axis of symmetry can either be horizontal or vertical. Hyperbolas also have asymptotes. An asymptote is a line that the branches of a hyperbola approach, but never cross. Asymptotes are denoted using dotted lines. See the diagram of a hyperbola with a horizontal axis of symmetry below.



The box in this diagram is intended only to help you determine the values of a, b, and c. It will not appear in all graphs of hyperbolas.

The equation of a hyperbola:

Vertical axis of Symmetry: $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

Horizontal axis of Symmetry: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

(h, k) is the center of the hyperbola. To find the center of the hyperbola, find the midpoint of the vertices. The variable a is the distance from the vertex to the center, b is the distance from the vertex to the asymptote, and c is the distance from the center to the point on the asymptote that the vertex connects with (it is also the distance from the center to the foci). The values of a, b, and c make a right triangle and the values of a, b, and c can be found using the Pythagorean Theorem: $a^2 + b^2 = c^2$.

Example 7: Write the equation of the hyperbola with vertices at (-6, 4) and (6, 4) and foci at
(-11, 4) and (11, 4).

Step 1: Determine the center of the hyperbola by finding the midpoint of the vertices.

$$(A)\left(\frac{-6+6}{2}\ ,\ \frac{4+4}{2}\right)$$

$$(B)\left(\frac{0}{2}\ ,\ \frac{8}{2}\right)$$

$$(C)\ (0,4)$$

Step 1A: Substitute the values from the vertices into the formula for the midpoint of a line.
Step 1B: Simplify the numerators of the fractions (-6 + 6 = 0 and 4 + 4 = 8).
Step 1C: The midpoint of the vertices is (0, 4).

The center is (0, 4) and h = 0, k = 4.

Step 2: Determine whether the hyperbola has a horizontal or vertical axis of symmetry. This can be accomplished by graphing the vertices and drawing the line between them. If the line is horizontal, the hyperbola has a horizontal axis of symmetry and if the line is vertical, the hyperbola has a vertical axis of symmetry. In this case, the hyperbola has a horizontal axis of symmetry. So, we will use the following equation.

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

Step 3: Determine the value of a and b. a is the distance from the vertex to the center, so a = 6. In order to determine the value of b, we are going to have to use the Pythagorean theorem and the value of c. The distance from the foci to the center is c, so c = 11. Now we can substitute a = 6 and c = 11 into the Pythagorean theorem to solve for b^2.

$$\begin{array}{cc} (A) & (B) \\ c^2 = a^2 + b^2 & 121 = 36 + b^2 \\ (11)^2 = (6)^2 + b^2 & 85 = b^2 \end{array}$$

Step 3A: Substitute a = 6 and c = 11 into the Pythagorean theorem.
Step 3B: Square the 6 and the 11 and solve for b^2.

Step 4: Substitute the appropriate values into the standard form for the equation of a hyperbola with a horizontal axis of symmetry $(h = 0, k = 4, a = 6, \text{and } b^2 = 85)$.

$$\begin{array}{cc} (A) & (B) \\ \frac{(x-0)^2}{(6)^2}-\frac{(y-4)^2}{85}=1 & \frac{x^2}{36}-\frac{(y-4)^2}{85}=1 \end{array}$$

Example 8: Write the equation of the hyperbola with vertices at (5, 3) and (5, -3) and foci at (5, 7) and (5, -7).

Step 1: Determine the center of the hyperbola by finding the midpoint of the vertices.

$$\begin{array}{ccc} (A) & (B) & (C) \\ \left(\frac{5+5}{2},\frac{3+-3}{2}\right) & \left(\frac{10}{2},\frac{0}{2}\right) & (5,0) \end{array}$$

Step 1A: Substitute the values from the vertices into the formula for the midpoint of a line.
Step 1B: Simplify the numerators of the fractions (5 + 5 = 10 and 3 + -3 = 0).
Step 1C: The midpoint of the vertices is (5, 0).

The center is (5, 0) and h = 5, k = 0.

Step 2: Determine whether the hyperbola has a horizontal or vertical axis of

symmetry. This can be accomplished by graphing the vertices and drawing the line between them. If the line is horizontal, the hyperbola has a horizontal axis of symmetry and if the line is vertical, the hyperbola has a vertical axis of symmetry. In this case, the hyperbola has a vertical axis of symmetry. So, we will use the following equation.

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Step 3: Determine the value of a and b. a is the distance from the vertex to the center, so a = 3. In order to determine the value of b, we are going to have to use the Pythagorean theorem and the value of c. The distance from the foci to the center is c, so c = 7. Now we can substitute a = 3 and c = 7 into the Pythagorean theorem to solve for b^2.

(A)	(B)
$c^2 = a^2 + b^2$	$49 = 9 + b^2$
$(7)^2 = (3)^2 + b^2$	$40 = b^2$

Step 3A: Substitute a = 3 and c = 7 into the Pythagorean theorem.
Step 3B: Square the 3 and the 7 and solve for b^2.
Step 4: Determine the equation of the hyperbola.

(A)	(B)
$\frac{(y-0)^2}{(3)^2} - \frac{(x-5)^2}{40} = 1$	$\frac{y^2}{9} - \frac{(x-5)^2}{40} = 1$

Step 4A: Substitute the values into the equation $(h = 5, k = 0, a = 3, \text{and } b^2 = 40)$. Step 4B: Square the value for a and simplify.

Classifying Conics:

To classify a conic, find the discriminant of the equation
$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
To find the discriminant use the following formula.

$$B^2 - 4AC$$

(a) If the discriminant is < 0 and A = C, the conic is a circle.
(b) If the discriminant is < 0 and $A \neq C$, the conic

(c) If the discriminant is > 0, the conic is a hyperbola.
(d) If the discriminant = 0, the conic is a parabola.

Example 9: Classify the conic.

$$3x^2 - 2xy - 4y^2 + 5x - 3y - 6 = 0$$

(1)	(2)	(3)
A = 3, B = -2	$(-2)^2 - 4(3)(-4)$	$4 - (-48)$
C = -4, D= 5		$4 + 48$
E = -3, F = -6		52

Step 1: Assign values for each letter.
Step 2: Find the discriminant by substituting into the formula $B^2 - 4AC$ Step 3: Square -2 (-2 x -2 = 4) and multiply 4(3)(-4) = -48. Change the problem from -(-48) to + 48 and add. The discriminant is 52.

Since the discriminant is positive, the conic is a hyperbola.

Example 10: Classify the conic.

$$x^2 + 6xy + 9y^2 - 15x - 24y + 16 = 0$$

(1)	(2)	(3)
$A = 1,\ B = 6$ $C = 9,\ D = -15$ $E = -24,\ F = 16$	$(6)^2 - 4(1)(9)$	$36 - 36$ 0

Step 1: Assign values for each letter.
Step 2: Find the discriminant by substituting into the equation $B^2 - 4AC$ Step 3: Square the 6 (6 x 6 = 36) and multiply 4(1)(9) = 36. Subtract 36 from 36 to get 0.

Since the discriminant is zero, the conic is a parabola.

Example 11: Classify the conic.

$$9x^2 + 9y^2 + 9x + 10y - 7 = 0$$

(1)	(2)
$9x^2 + 0xy + 9y^2 + 9x + 10y - 7 = 0$	$A = 9, B = 0$ $C = 9, D = 9$ $E = 10, F = -7$
(3)	**(4)**
$0^2 - 4(9)(9)$	$0 - 324$ -324

Step 1: Rewrite the equation in standard form.
Step 2: Assign values for each letter.
Step 3: Find the discriminant by substituting into the equation $B^2 - 4AC$ Step 4: Square 0 (0 x 0 = 0) and multiply 4(9)(9) = 324. Subtract 324 from 0 to get -324.

Since the discriminant is negative and A = C, the conic is a circle.

Example 12: Classify the conic.

$$6x + y^2 - 12y + 2x^2 = 0$$

(1)	(2)
$2x^2 + 0xy + y^2 + 6x - 12y + 0 = 0$	$A = 2, B$ $C = 1, D$ $E = -1$
(3)	**(4)**
$(0)^2 - 4(2)(1)$	$0 -$ -8

Step 1: Rewrite the equation in standard form.
Step 2: Assign values for each letter.
Step 3: Find the discriminant by substituting into the equation $B^2 - 4AC$ Step 4: Square 0 (0 x 0 = 0) and multiply 4(2)(1) = 8. Subtract 8 from 0 to get -8.

Since the discriminant is negative and $A \neq C$, t

Complex Numbers
Descartes (a mathematician and philosopher) called the square roots of negative numbers imaginary numbers, in contrast to the numbers everyone understood, which he called "real numbers." Another mathematician (Euler) used the symbol "i" to denote imaginary numbers.

Definition: $i = \sqrt{-1}$

It is assumed that imaginary numbers have

the same properties as other numbers. They can be added, subtracted, multiplied, and divided.

Example 1: Simplify.

$$\sqrt{-25}$$
$$(1)\ \sqrt{-25} = \sqrt{25 \cdot -1}$$
$$(2)\ \sqrt{25 \cdot -1} = \sqrt{25} \cdot \sqrt{-1}$$
$$(3)\ \sqrt{25} = \pm 5,\ \sqrt{-1} = i$$
$$(4)\ \sqrt{-25} = \pm 5 \cdot i = \pm 5i$$

Step 1: The square root of -25 can be rewritten as the square root of the product of 25 and -1.
Step 2: The square root of 25 times -1 can be rewritten as the product of the square root of 25 and the square root of -1.
Step 3: The square root of 25 equals +5 or -5 (5 x 5 = 25 and -5 x -5x= 25). The square root of -1 equals i (see the definition of i above).
Step 4: The square root of -25 equals +5 or -5 times i.

The correct answer is $\pm 5i$.

Adding and Subtracting Complex Numbers:

A complex number is a number in the form a + bi, where a and b are real numbers and i = the square root of -1. Adding and subtracting complex numbers is similar to collecting like terms in that only imaginary numbers can be added to or subtracted from imaginary numbers. For example, 2i + 3i = 5i, but 2 + 3i cannot be simplified.

Example 2: Combine the complex numbers.
(3 + 4i) + (-7 -6i)

(1) 3 + 4i - 7 - 6i
(2) (3 -7) + (4i - 6i)
(3) -4 + -2i or -4 - 2i

Step 1: Write the complex numbers out as one long problem.
Step 2: Combine the real numbers in one set of parentheses, and the imaginary numbers in another set of parentheses.
Step 3: 3 - 7 = -4 and 4i - 6i = -2i, so (3 + 4i) + (-7 -6i) equals -4 + -2i. -4 + -2i can also be written
-4 - 2i.

The correct answer is -4 - 2i.

Example 3: Combine the complex numbers.
(10 + 7i) + (4 - 9i) - (2 - 4i)

(1) 10 + 7i + 4 - 9i - 2 + 4i
(2) (10 + 4 - 2) + (7i - 9i + 4i)
(3) 12 + 2i

Step 1: First, distribute the subtraction symbol through the terms in the third set of parentheses (this involves changing the sign of every number in the set of parentheses: 2 becomes -2 and -4i becomes +4i).Then write the complex numbers out as one long problem.
Step 2: Combine the real numbers in one set of parentheses and the imaginary numbers in another set of parentheses.
Step 3: 10 + 4 - 2 = 12 and 7i - 9i + 4i = 2i, so (10 + 7i) + (4 - 9i) - (2 - 4i) equals 12 + 2i.

The correct answer is 12 + 2i.

Multiplying Complex Numbers:

There are two types of problems which require multiplying complex numbers. One of those types is multiplying a complex number by a constant factor. The other type is multiplying a complex number by another complex number. When multiplying complex numbers, it is important to remember the following definition.

Definition: $i^2 = -1$

Proof: $i = \sqrt{-1}$

$$i^2 = \left(\sqrt{-1}\right)^2 = -1$$

Example 4: Multiply. 2i(3 + 7i)

(1) $2i \cdot 3 + 2i \cdot 7i$

(2) $6i + 14i^2$

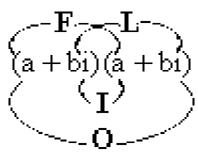
(3) $6i + 14(-1)$

(4) $-14 + 6i$

Step 1: Distribute the 2i to each term in the parentheses. This involves multiplying each term in the parentheses by 2i.
Step 2: To multiply an imaginary number by a real number or by another imaginary number, first multiply the real numbers. Then multiply the i's. 2i x 3 = 2 x 3 x i = 6i and $2i \cdot 7i = 2 \cdot 7 \cdot i \cdot i = 14i^2$ Step 3: It is known that $i^2 = -1$. Substitute -1 into the problem in place of the i^2. Step 4: Multiply 14 by -1. Write this number first because complex numbers are always written with the imaginary number last (a + bi). Then add the 6i to the end.

The correct answer to 2i(3 + 7i) is -14 + 6i.

Multiplying a complex number by a complex number is much like using the FOIL Method to multiply a binomial by a binomial. The FOIL Method states that the **F**irst terms of each complex number are multiplied, then the **O**uter terms of each complex number are multiplied, next the **I**nner terms of the complex numbers are multiplied, and finally, the **L**ast terms of the complex numbers are multiplied. See the diagram below.



Example 5: Multiply. (2 + 5i)(3 + 4i)

(1)

F $2 \cdot 3 = 6$

O $2 \cdot 4i = 8i$

I $5i \cdot 3 = 15i$

L $5i \cdot 4i = 20i^2$

(2) $6 + 8i + 15i + 20i^2$ $\quad 6+$

(4) $6 + 23i + 20(-1)$

(5) $6 + 23i - 20$ $\quad$ -

Step 1: Use the FOIL Method to multiply the complex numbers. 2 and 3 are the **F**irst terms in the complex numbers, 2 and 4i are the **O**uter terms, 5i and 3 are the **I**nner terms, and 5i and 4i are the **L**ast terms of the complex numbers.
Step 2: Add the products acquired from using the FOIL Method.
Step 3: 8i and 15i can be added together to make 23i.
Step 4: Substitute -1 in place of i^2. Step 5: Multiply 20 by -1 to get -20.
Step 6: 6 + -20 = -14, so (2 + 5i)(3 + 4i) = -14 + 23i.

The correct answer to (2 + 5i)(3 + 4i) is -14 + 23i.

Simplifying Expressions with Complex Numbers:

It is necessary to know the order of operations when simplifying expressions. The **order of operations** is just what it sounds like: the order in which one computes the operations in an expression. Here is the order of operations:

(1) Parentheses, Brackets, and Braces
(2) Exponents or Roots
(3) Multiply or Divide in order from left to right
(4) Add or Subtract in order from left to right

Here are a few helpful hints for using the order of operations. First, remember to

complete all operations of one type before moving on to the next type (for example, complete all multiplication and division before moving on to addition or subtraction). Second, remember that when working the multiplication or division move from the left to the right (for example, 2 x 6 ÷ 3. In this case, you would multiply (2 x 6) first because the multiplication is the first operation when reading from the left to the right and then divide by 3 to get 4). Finally, addition and subtraction work the same way as multiplication and division - from the left to the right (for example, 10 - 6 + 2. In this case, you would subtract 6 from 10 first, then add the 2 to get 6).

Example 6: Simplify the expression.

5(1 + 3i) - 8(2 - 4i)

(1) 5(1) + 5(3i) - 8(2 - 4i)
(2) 5 + 15i + -8(2 - 4i)
(3) 5 + 15i + -8(2) - -8(4i)
(4) 5 + 15i + -16 + 32i
(5) (5 + -16) + (15i + 32i)
(6) -11 + 47i

Step 1: Following the order of operations, we would have to complete the operations inside the parentheses first. It is impossible to add 1 + 3i (because they are not like terms) and it is impossible to subtract 2 - 4i (they are not like terms, either). So, we must move to the next possible operation. That operation is multiplication because there are no exponents in the problem. Distribute the 5 to each term in the first set of parentheses. This involves multiplying each term in the first set of parentheses by 5.
Step 2: Multiply 5 by 1 and 5 by 3i. Then change the subtraction symbol to an addition symbol and make the 8 negative. The 8 was always negative (it had a negative sign to its left), this step just makes us realize the negative for our next step.
Step 3: Distribute the -8 to each term in the parentheses. This involves multiplying each term in the parentheses by -8.
Step 4: Write out the new problem.
Step 5: Combine the real numbers in one set of parentheses and combine the imaginary numbers in another set of parentheses.
Step 6: 5 + -16 = -11 and 15i + 32i = 47i, so 5(1 + 3i) - 8(2 - 4i) = -11 + 47i.

The correct answer is -11 + 47i.

Solving Equations that Involve Imaginary Numbers:

To solve equations that involve imaginary numbers we follow the steps for solving equations. The goal is to isolate the variable on one side of the equal sign. Here is an example.

Example 7: Solve the following equation.

$$-4x^2 + 2 = 146$$

(1)

$$\begin{array}{r} -4x^2 + 2 = 146 \\ -2 \quad -2 \\ \hline -4x^2 = 144 \end{array}$$

(2)

$$\frac{-4x^2}{-4} = \frac{144}{-4}$$

$$x^2 = -36$$

(3)

$$\sqrt{x^2} = \sqrt{-36}$$

$$x = \pm\sqrt{-36}$$

(4)

$$x = \pm\sqrt{-36}$$

$$x = \pm\sqrt{36 \cdot -1}$$

$$x = \pm\sqrt{36} \cdot \sqrt{-1}$$

$$x = \pm 6 \cdot i$$

$$x = \pm 6i$$

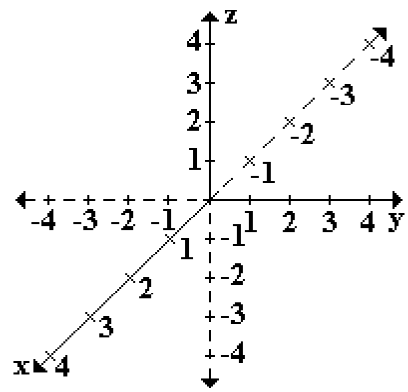
Step 1: Subtract 2 from each side of the equation. This begins the process of isolating the variable.
Step 2: Divide each side of the equation by -4. This will isolate the variable on one side of the equal sign.
Step 3: Take the square root of each side of the equal sign. Since squaring and taking the square root are opposite operations, we are left with an isolated x on one side of the equal sign. Since every number has a positive and negative square root, the answer should now read $\pm\sqrt{-36}$. This shows that there are 2 answers to the problem.
Step 4: Use the steps in **Example 1** to determine the square roots of -36.

The correct answers are 6i and -6i.

Three Dimensional Space

Points of the form (x, y), called an ordered pair, can be plotted in a rectangular or two-dimensional coordinate system. Points of the form (x, y, z), called an ordered triple, can be plotted in a three-dimensional coordinate system.

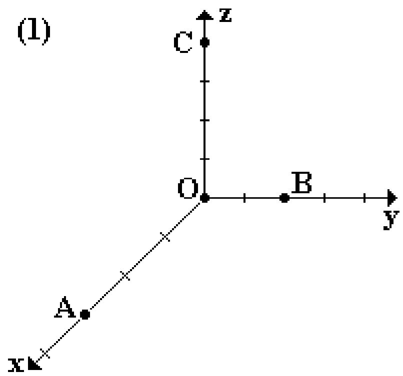
As indicated in the diagram, this new coordinate system has three axes. Note that the x- and y-axes are not arranged in the same configuration as in the rectangular coordinate system. Also note the location of positive and negative numbers along each of the axes. The origin (0, 0, 0) is located at the intersection of the three axes.



Plotting a point in three dimensional space:

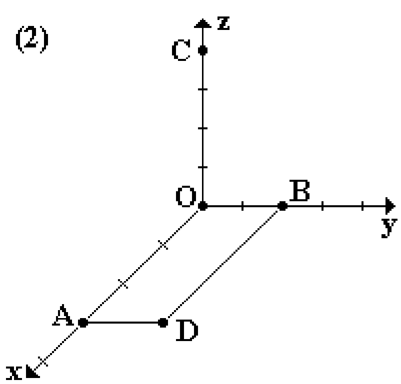
A rectangular box is often created to plot a point in a three-dimensional coordinate system. The box helps to give a sense of a third dimension in a drawing that is two-dimensional. The vertices of the box are found, starting with the origin, and then segments joining these vertices are drawn. The final vertex of the box shows the position of the given point. The following example shows how to plot the point (3, 2, 4).

(1)



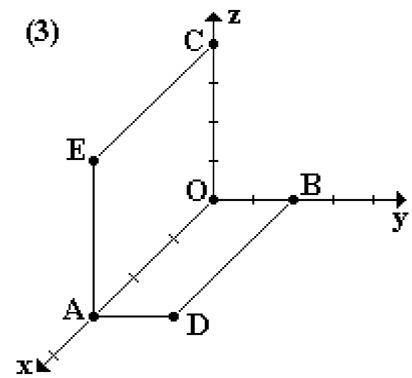
Step 1: From the origin, O, follow along the x-axis for 3 units to locate point A, (3, 0, 0) because the x-coordinate of the given point is 3. From the origin, locate point B, 2 units along the y-axis, at (0, 2, 0) because the y-coordinate is 2. From the origin, locate point C, 4 units along the z - axis at (0, 0, 4) because the z-coordinate is 4. Note that any point that lies on an axis will have two coordinate values equaling 0.

(2)

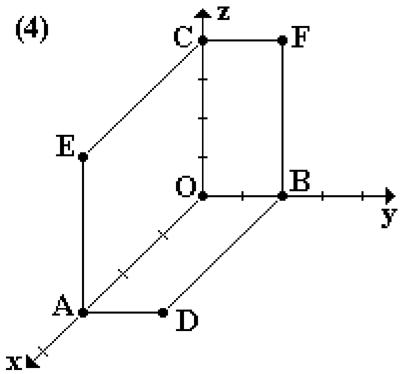


Step 2: Now draw the bottom of the box. From point A, draw a segment 2 units long that is parallel to the y - axis (since the y - coordinate of the given point is 2) to locate point D. The coordinates of D are (3, 2, 0). Connect the points B and D to form the bottom.

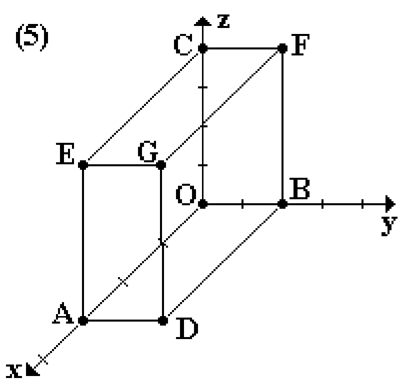
(3)



Step 3: Next draw the left side of the box. From point A, draw a segment 4 units long that is parallel to the z-axis (since the z-coordinate is 4) to locate point E. The coordinates of E are (3, 0, 4). Connect points C and E to form the left side.



Step 4: Next draw the back of the box. From point B, draw a segment 4 units long that is parallel to the z-axis (since the z-coordinate is 4) to locate point F. The coordinates of F are (0, 2, 4). Connect points C and F to finish the back side.



Step 5: Complete the box by drawing the top. From point E, draw a segment 2 units long that is parallel to the y-axis (since the y-coordinate is 2) to locate point G. Connect points F and G to complete the box. The coordinates of G are (3, 2, 4).

Determining the fourth vertex of a rectangle:

Consider the rectangle that is located at the top of the above example. Its coordinates are E(3, 0, 4), C (0, 0, 4), F(0, 2, 4) and G(3, 2, 4). Note that all of the z-coordinates are equal to 4 because the top is located 4 units up from the origin. There are two different x-coordinates, x = 3 and x = 0. Each x-value appears in two ordered triples. There are two different y-coordinates, y = 0 and y = 2. Each y-value appears in two ordered triples. To summarize:

- One coordinate is the same in each of the four vertices of a rectangle that is parallel to one of the axes.
- The values of each of the other two coordinates take on two different values, each one appearing in two different pairs of ordered triples.

The above observations will be used to solve the following example problems.

Example 1: The points (3, 2, 4), (3, 2, 8), and (3, 6, 4) make up three of the vertices of a rectangle. What is the fourth point?

To solve this problem, first note that all three given points have the same x-value, x = 3. To form the shape of a rectangle the fourth point must have the same x-value as the other points, so x = 3. In addition, to form a rectangle, the y- and z-values of all four points must come in pairs. There are already two occurrences of y = 2, but only one of y = 6, so the fourth point must have y = 6. There is already one pair of points with z = 4, so the fourth point must have z = 8. Thus the fourth point is (3, 6, 8).

Example 2: The points (-3, 2, 5), (6, 2, 5), and (-3, 2, 4) make up three of the vertices of a rectangle. What is the fourth point?

To solve this problem, first note that all three given points have the same y-value, y = 2. To form the shape of a rectangle the fourth point must have the same y-value as the other points, so y = 2. In addition, to form a rectangle, the x- and z-values of all four points must come in pairs. There are already two pairs of x = -3, but only one of x

= 6, so the fourth point must have x = 6. There is already one pair of points with z = 5, so the fourth point must have z = 4. Thus the fourth point is (6, 2, 4).

Example 3: The points (0, 1, -6), (3, 1, -6), and (3, 2, -6) make up three of the vertices of a rectangle. What is the fourth point?

To solve this problem, first note that all three given points have the same z-value, z = -6. To form the shape of a rectangle the fourth point must have the same z-value as the other points, so z = -6. In addition, to form a rectangle, the x- and y-values of all four points must come in pairs. There is already one pair values of x = 3, but only one of x = 0, so the fourth point must have x = 0. There is already one pair of ordered triples with y = 1, so the fourth point must have y = 2. Thus the fourth point is (0, 2, -6).

Determining a fourth vertex of a cube:

A cube is a solid in which all six of its faces (sides) have the shape of a square. Therefore the lengths of each of its edges will all be the same.

Example 4: A cube has the following vertices: (6, 19, 5), (6, 12, 5), and (13, 12, 5). Which point is also a vertex of the cube?

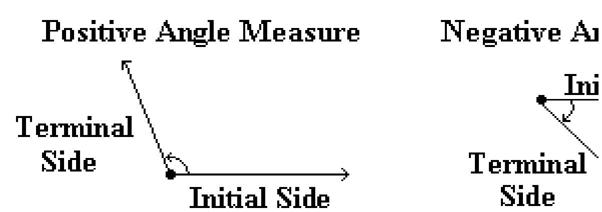
A. (13, 12, 8)
B. (13, 19, 8)
C. (6, 12, 12)
D. (13, 11, 12)

To solve this problem, note that since the ordered triples represent vertices of a cube, the lengths of the sides of cube must be the same. All three of the given vertices have the same z coordinate, so they form 3 of the edges of one side of the cube. The distance between vertices, (6, 19, 5) and (6, 12, 5) is found by subtracting 19 - 12 = 7. To find this length subtract the pair of non-matching coordinates. The distance between vertices, (6, 19, 5) and (13, 12, 5), is found by subtracting 13 - 6 = 7. In fact, the distance between any two vertices that form an edge of this cube will be 7.

To find the missing vertex, check each of the answer options to determine which one provides an edge that has a length of 7. If Choice A, (13, 12, 8), is chosen, pick the third given vertex, (13, 12, 5), to use as a test point since two out of three coordinates match. Subtract the different coordinates to get 8 - 5 = 3. This is not the desired answer, so check Choice B, (13, 19, 8). Because none of the given vertices share two out of three coordinates with Choice B, it is probably not the answer. Check Choice C, (6, 12, 12), and compare if with the vertex (6, 12, 5). The distance between these two points is 12 - 5 =7 so choice C, (6, 12, 12), is also a vertex of the cube.

Radians

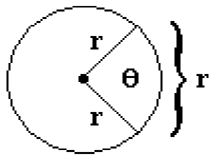
In trigonometry, it is helpful to define an angle which is formed by a ray that is rotated in a plane about its endpoint. The original position of the ray is called the initial side. The terminal side is the position of the ray at the end of the rotation. If the terminal ray moves in a counterclockwise direction, the angle has a positive measure. If the terminal ray moves in a clockwise direction, the angle has a negative measure.



Radians:

A radian is defined to be the measure of a central angle (an angle whose vertex lies at the center of a circle) of a circle in which the sides of the angle intercept an arc that is equal in length to the radius, r, of the circle.

The central angle is often represented by the Greek letter theta, θ



In the above diagram, theta has a measure of 1 radian. The circumference of any circle is given by the following formula.

$$C = 2\pi r$$

To determine the number of radians in one revolution of the circle (one trip around the circle), divide the circumference by r, the length of each radian, to get the number of radians in 1 revolution.

$$\frac{2\pi r}{r} = 2\pi \text{ radians in 1 revolution}$$

Since one revolution is also equivalent to 360º , then $2\pi \text{ radians} = 360°$. To get a sense of the degree measure of a radian, note the following:

$$(1)\ 2\pi \text{ radians} = 360°$$

$$(2)\ \frac{2\pi \text{ radians}}{2\pi} = \frac{360°}{2\pi}$$

$$(3)\ 1 \text{ radian} = \frac{180°}{\pi}$$

$$(4)\ 1 \text{ radian} = \frac{180°}{3.1416}$$

$$1 \text{ radian} \approx 57.3°$$

Step 1: To solve for 1 radian, write out the equation $2\pi \text{ radians} = 360°$. Step 2: Divide both sides by 2π Step 3:

The 2π in the numerator and denominator of the left fraction cancel out. $360° \div 2 = 180°$.

Step 4: Since pi is approximately equal to 3.1416, an approximate degree value for one radian can be found to equal 57.3º .

Thus one radian is just less than 60º .

Note that angles that are measured in degrees are denoted with the degree symbol (º), while angles measured in radians do not have to be labeled.

Thus, π radians can be written as π.

Determining the complement and supplement of an angle:

The sum of two complementary angles is 90º or $\frac{\pi}{2}$. To find the complement of an angle, subtract the given angle from 90º or $\frac{\pi}{2}$.

Example 1: What is the complement of $\frac{\pi}{3}$?

(1)	(2)	(3)
$\frac{\pi}{2} - \frac{\pi}{3}$	$\frac{\pi}{2} \cdot \frac{3}{3} - \frac{\pi}{3} \cdot \frac{2}{2}$	$\frac{\pi}{6}$
	$\frac{3\pi}{6} - \frac{2\pi}{6}$	

Step 1: To find the complement of an angle, subtract the given angle from $\frac{\pi}{2}$. Step 2: Find the least common denominator to be 6 and rewrite each fraction.
Step 3: Subtract the fractions to get the answer.

The answer is $\frac{\pi}{6}$

Determining the Supplement of an Angle:

The sum of two supplementary angles is 180º or π. To find the supplement of an angle, subtract the given angle from 180º or π.

Example 2: Find the supplement of $\frac{3\pi}{5}$.

(1) $\pi - \frac{3\pi}{5}$

(2) $\frac{\pi}{1} \cdot \frac{5}{5} - \frac{3\pi}{5}$

$\frac{5\pi}{5} - \frac{3\pi}{5}$

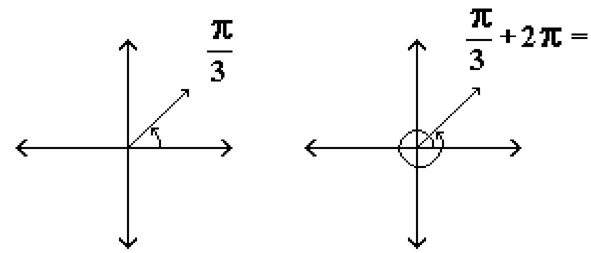
(3) $\frac{2\pi}{5}$

Step 1: To find the supplement of an angle, subtract the given angle from π.
Step 2: Find the least common denominator to be 5 and rewrite π.
Step 3: Subtract the fractions to get the answer.

The answer is $\frac{2\pi}{5}$

Coterminal angles:

An angle that is in standard position is an angle whose vertex is positioned at the origin and whose initial ray lies on the positive x-axis. Angles that are in standard position and share the same terminal ray are called coterminal angles. Any angle has an infinite number of coterminal angles. To find a coterminal angle of a given angle, simply add or subtract multiples of 360º or 2 pi from the given angle. Each multiple of 360º or 2 pi takes the terminal ray of the given angle and makes it complete one more revolution about the origin such that the terminal ray will end up in the same position it was before it was rotated. (Note that the ray can move in a counterclockwise or positive direction as well as a clockwise or negative direction.) Also remember that 360º $= 2\pi$



Example 3:

Find a coterminal angle for $\theta = \frac{\pi}{6}$.

(1) $\frac{\pi}{6} + 2\pi$

$\frac{\pi}{6} + 2\pi \cdot \frac{6}{6}$

$\frac{\pi}{6} + \frac{12\pi}{6}$

(2) $\frac{13\pi}{6}$

Step 1: To find a coterminal angle of a given angle, simply add multiples of 2π Find the least common denominator to be 6 and rewrite π Step 2: Add the fractions to get the answer.

A coterminal angle is $\frac{13\pi}{6}$

Converting from radians to degrees:

To convert the measurement of any angle (including negative angles) from radian measure to degree measure, multiply the radian measure by $\frac{180°}{\pi}$.

Example 4: Rewrite $\frac{2\pi}{5}$ in degrees.

$$\frac{2\pi}{5} \cdot \frac{180°}{\pi} = 72°$$

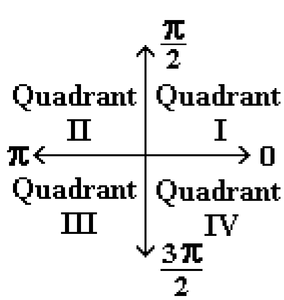
Solution: Multiply the given angle by $\frac{180°}{\pi}$. The pi values will cancel since there is a pi in the numerator and denominator. Then simplify to get 72º .

If the angle was changed to a negative value, the answer would be negative.

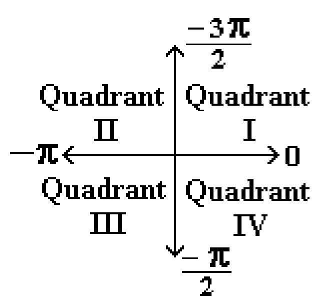
Locate the quadrant in which the

terminal ray of an angle lies:

The x- and y-axes of a rectangular coordinate plane divide the plane into four quadrants as labeled below. The terminal ray of an angle in standard position either lies on one of the axes or in one of the four quadrants. The numbers marked on the axes show the radian measure of an angle whose terminal ray lies on that portion of the axis.



Positive Angles



Negative Angles

Example 5:

In which quadrant is $-\frac{2\pi}{5}$ located?

The measure of the given angle lies between

0 and $-\frac{\pi}{2}$

Answer: quadrant IV

Example 6:

In which quadrant is $\frac{14\pi}{3}$ located?

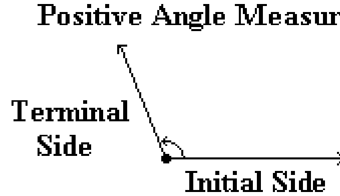
Since the measure of the given angle is greater
terminal ray must rotate before it is positioned
has a measure in the interval $0 \le x < 2\pi$. Rewri
$4\frac{2\pi}{3}$. In this new form, it is easier to see that t
$2 \cdot 2\pi = 4\pi$. Subtract $4\frac{2\pi}{3} - 4\pi$ to get a cotermi
$\frac{2\pi}{3}$ lies in the second quadrant because it has
$\frac{14\pi}{3}$ also lies in the second quadrant.

Answer: quadrant II

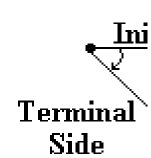
Angles

In trigonometry, it is helpful to define an angle that is formed by a ray that is rotated in a plane about its endpoint. The original position of the ray is called the <u>initial side</u>. The <u>terminal side</u> is the position of the ray at the end of the rotation. If the terminal ray moves in a counterclockwise direction, the angle has a positive measure. If the terminal ray moves in a clockwise direction, the angle has a negative measure.

Positive Angle Measure



Negative A



Radians:

A <u>radian</u> is defined to be the measure of a

central angle (an angle whose vertex lies at the center of a circle) of a circle in which the sides of the angle intercept an arc that is equal in length to the radius, r, of the circle.

To rewrite degrees in radians, use the formula below:

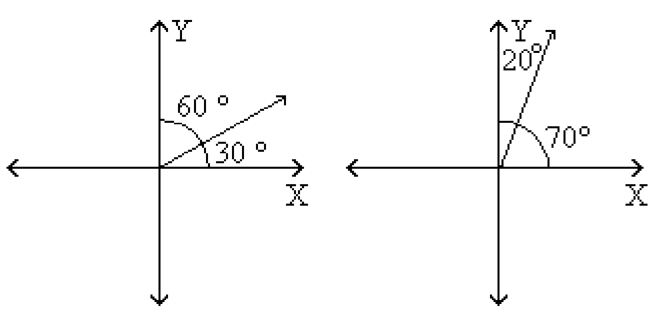
$$\text{degrees} \cdot \frac{\pi}{180} = \text{radians}$$

Example 1: Write 90º in radians.

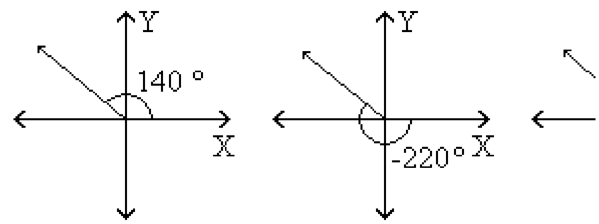
$$90^\circ \cdot \frac{\pi}{180} = \frac{\pi}{2}$$

The answer is $\frac{\pi}{2}$

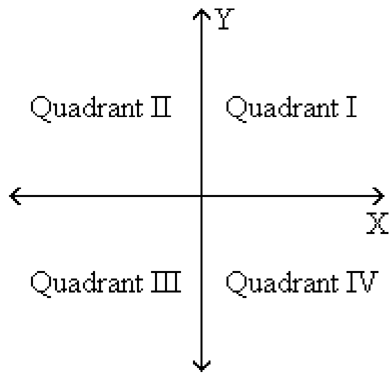
<u>Complementary angles</u> are two angles in which the sum of their measures is 90º . The complementary angle of a 30º angle is 60 º because 30º + 60º = 90º . The complementary angle of a 20º angle is a 70º angle because 20º + 70º = 90º .



<u>Coterminal angles</u> "end" at the same place. Coterminal angles differ by the number of revolutions. The figures below illustrate three coterminal angles:

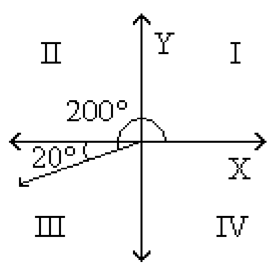


In order to draw the angle, it is easiest to find the reference angle. The <u>reference angle</u> is the measure of the acute angle (an acute angle is an angle less than 90 degrees) between the terminal side (end) and the x-axis. Here are the rules, but it is probably easier to sketch the angle and mathematically figure it out.



If the angle ends in quadrant I, the reference angle is equal to the angle.
If the angle ends in quadrant II, the reference angle is (180º - the angle).
If the angle ends in quadrant III, the reference angle is (the angle - 180º).
If the angle ends in quadrant IV, the reference angle is (360º - the angle).

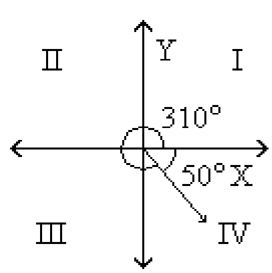
Example 2: Find the reference angle for 200º .



Solution: The terminal side of the angle is in the third quadrant, so we subtract 180º from the measure of the angle.

The reference angle for 200º is: 200º - 180º = 20º .

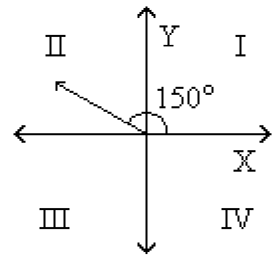
Example 3: Find the reference angle for 310º .



Solution: The terminal side of the angle is in the fourth quadrant, so subtract the measure of the angle from 360º .

The reference angle for 310º is: 360º - 310º = 50º .

Example 4: Find the reference angle for 150º .



Solution: Since the terminal side of the angle is in quadrant II, we take 180º - 150º = 30º .
The reference angle for 150º is: 30º .

Now that we know how to find reference angles, we can determine coterminal angles. Coterminal angles must have the same terminal side.

Example 5: Find two angles that are coterminal.

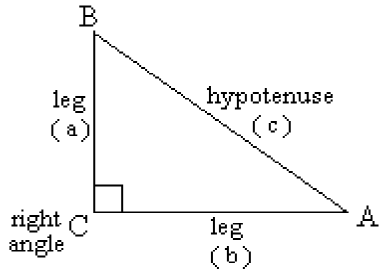
A. 45 º and 300 º
B. 25 º and 155 º
C. -310 º and 50 º
D. 75 º and -225 º

Only angles with the same reference angle have the possiblity of being coterminal. Therefore, we check choices Band C since both have identical reference angles. The angles in choice C also have the same terminal side, so the answer is C. Both angles have a reference angle of 50º and the same terminal side.

With the help of reference angles, we can find the trigonometric ratios. (It may be helpful at this point to review the Pythagorean Theorem).

The Pythagorean Theorem can be used to find the length of a side of a right triangle when the lengths of the other two sides are known. When the measures of a side and an acute angle of a right triangle are known and

we need to compute the length of another side or the measure of another angle, trigonometry makes it possible.



$a^2 + b^2 = c^2$

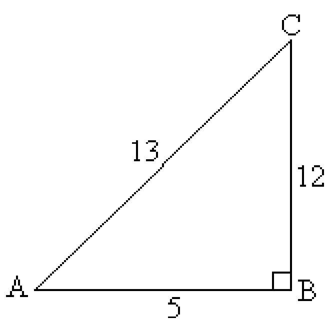
The basic trigonometric ratios are defined below for angle A of $\triangle ABC$

$$\text{Sine of } \angle A = \frac{\text{length of side opposite } \angle A}{\text{length of hypotenuse}} = \frac{a}{c}$$

$$\text{Cosine of } \angle A = \frac{\text{length of side adjacent } \angle A}{\text{length of hypotenuse}} = \frac{b}{c}$$

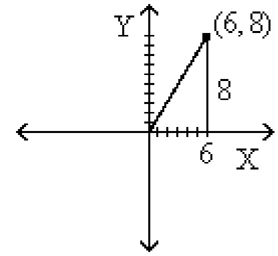
$$\text{Tangent of } \angle A = \frac{\text{length of side opposite } \angle A}{\text{length of side adjacent } \angle A} = \frac{a}{b}$$

Example 6: Find sin A, cos A, and tan A for $\triangle ABC$



The answer is: sin A = 12/13, cos A = 5/13, and tan A = 12/5.

Example 7: Find the sine if (6, 8) is on the terminal side of the angle.



Step 1: Sketch the graph, plot the point (6, 8), and sketch the ray from the origin to the point (6, 8). Then, sketch the line down to the x-axis forming a right triangle.
Step 2: Determine the length of the hypotenuse using the Pythagorean Theorem:

(1) $6^2 + 8^2 = c^2$

(2) $36 + 64 = c^2$

(3) $100 = c^2$

(4) $\sqrt{100} = \sqrt{c^2}$

(5) $10 = c$

$$\text{Sin}A = \frac{\text{length of side opposite} \angle A}{\text{length of hypotenuse}}$$

Step 3: $\text{Sin}A = \frac{8}{10} = \frac{4}{5}$

Example 8: Evaluate the function.
sec 300°

Solution: We know that sec = 1/cos. We can use this information to determine the value of sec 300° . The cosine of 300° is -1/2 because 300° falls in the fourth quadrant.

$$\sec 300° = \frac{1}{-\frac{1}{2}} = -2$$

Answer: -2

Trigonometric Identities

A trigonometric identity is a statement that is true for all angle measures. Angle measures can be represented by variables or by the Greek letter theta θ An example of an identity is $\sin^2\theta + \cos^2\theta = 1$. Note that it is not an equation to be solved. Any value substituted in for theta will make the

statement true. For example:

If $\theta = 60°$, then...

$\sin^2\theta + \cos^2\theta = 1$	Replace θ by $60°$
$\sin^2 60° + \cos^2 60° = 1$	Recall that $\sin 60° = \frac{\sqrt{3}}{2}$ and c
$\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1$	$\left(\frac{\sqrt{3}}{2}\right)^2 = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{3}{4}$ and $\left(\frac{1}{2}\right)$
$\frac{3}{4} + \frac{1}{4} = 1$	True Statement
$\frac{4}{4} = 1$	
$1 = 1$	

Type of Identity with Examples

Reciprocal Identities:

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \quad \tan\theta = \frac{\sin\theta}{\cos\theta} \quad \csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \quad \csc^2\theta = \frac{1}{\sin^2\theta}$$

Pythagorean Identities:

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \cot^2\theta = \csc^2\theta$$

There are many different types of trigonometric identities which are often summarized in the trigonometry section of a textbook. Some of the general categories are listed below as well as a specific identity for that category:

Cofunction Identities:

$$\cos\left(\frac{\pi}{2}-x\right)=\sin x \qquad \sin\left(\frac{\pi}{2}-x\right)=\cos x$$

$$\cot\left(\frac{\pi}{2}-x\right)=\tan x$$

Negative Identities:

$$\sin\left(^{-}\theta\right)={}^{-}\sin\theta \qquad \cos\left(^{-}\theta\right)=\cos\theta$$

$$\cot\left(^{-}\theta\right)={}^{-}\cot\theta \qquad \tan\left(^{-}\theta\right)={}^{-}\tan\theta$$

$$\csc\left(^{-}\theta\right)={}^{-}\csc\theta \qquad \sec\left(^{-}\theta\right)=\sec\theta$$

Sum and Difference Identities:

$$\cos\left(x+y\right)=\cos x\cos y-\sin x\sin y$$
$$\sin\left(x+y\right)=\sin x\cos y+\cos x\sin y$$
$$\cos\left(x-y\right)=\cos x\cos y+\sin x\sin y$$
$$\sin\left(x-y\right)=\sin x\cos y-\cos x\sin y$$

Double Angle Identities

$$\sin 2\theta = 2\sin\theta\ \cos\theta$$
$$\cos 2\theta = \cos^2\theta-\sin^2\theta$$

Note that to correctly and more easily solve trigonometric identity problems requires memorization of the basic identities.

Example 1: Complete the following identity.

$$\frac{\cos\theta}{\sin\theta}= \underline{\qquad\qquad}$$

There is a reciprocal identity that states that $\frac{\cos\theta}{\sin\theta}=\cot\theta$.

Example 2: Simplify the following expression.

$$\frac{\sin\left(^{-}x\right)}{\tan\left(^{-}x\right)}$$

(1)

$$\frac{\sin\left(^{-}x\right)}{\tan\left(^{-}x\right)}=\frac{^{-}\sin x}{^{-}\tan x}$$

(2)

$$\frac{^{-}\sin x}{-\frac{\sin x}{\cos x}}$$

(4)

$$\frac{^{-}\sin x}{1}\cdot-\frac{\cos x}{\sin x}$$

(5)

$$\cos x$$

Step 1: Use the Negative identities that state that sin (- x) = - sin x and tan (-x) = - tan x. Substitute the equivalent expressions to rewrite the original fraction and obtain the fraction on the right.
Step 2: It is often helpful to rewrite all of the expressions in terms of sin x and cos x to make it easier to determine what the next step should be. In this problem, the reciprocal identity, tan x = sin x/cos x, was used to rewrite the tan x term in Step 1 and create the new complex fraction of Step 2.
Step 3: The complex fraction of Step 2 can be rewritten to use the conventional division sign, ÷ , to help determine the next step.
Step 4: A division problem containing fractions can be rewritten as a multiplication problem by multiplying the first term of the division problem by the reciprocal of the second term. The - sin x is rewritten as a fraction by making the denominator one (-sin x/1).
Step 5: The sin x terms in the numerator and denominator of the fraction divide out. The product of two negative quantities is positive, so the product shown in Step 4 simplifies to cos x.

The answer is cos x.

Example 3: Simplify the following.

$$\sin x\ \sin\left(\frac{\pi}{2}-x\right)+\cos x\ \cos\left(\frac{\pi}{2}-x\right)$$

(1) sin x cos x + cos x sin x
(2) sin x cos x + sin x cos x = 2sin x cos x

Step 1: Use the cofunction identity, $\sin\left(\frac{\pi}{2} - x\right) = \cos x$ to substitute $\cos x$ in place of $\sin\left(\frac{\pi}{2} - x\right)$ and the cofunction identity $\cos\left(\frac{\pi}{2} - x\right) = \sin x$ to substitute $\sin x$ in place of $\cos\left(\frac{\pi}{2} - x\right)$.

Step 2: Rewrite $\cos x \sin x$ as $\sin x \cos x$. This will allow you to add $\sin x \cos x$ and $\sin x \cos x$ to get $2\sin x \cos x$.

The final answer is 2sin x cos x.

Determine the trigonometric functions of an angle when given the sides of a right triangle:

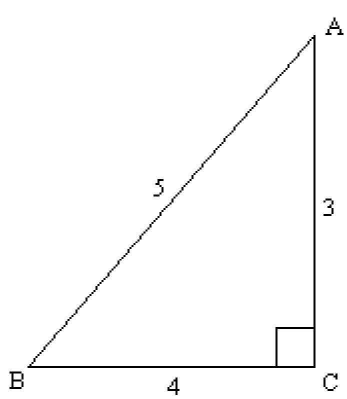
To solve right triangle trigonometry problems, the following ratios need to be used. Note that "opp" represents the side of the triangle opposite the given angle, A, "adj" represents the side of the triangle adjacent to the given angle, A, and "hyp" represents the hypotenuse of the right triangle.

$$\sin A = \frac{\text{opp}}{\text{hyp}} \qquad \csc A = \frac{\text{hyp}}{\text{opp}}$$

$$\cos A = \frac{\text{adj}}{\text{hyp}} \qquad \sec A = \frac{\text{hyp}}{\text{adj}}$$

$$\tan A = \frac{\text{opp}}{\text{adj}} \qquad \cot A = \frac{\text{adj}}{\text{opp}}$$

Example 4: Find the sine, cosine, tangent, cosecant, secant, and cotangent for angle ABC.



First identify the specific sides of the right triangle with respect to the given angle. In this problem, the side opposite angle ABC has length 3, so opp = 3. The side adjacent to angle ABC has length 4, so adj = 4 and the hypotenuse has length 5, so hyp = 5. The ratios are given below.

$$\sin\angle ABC = \frac{\text{opp}}{\text{hyp}} = \frac{3}{5} \qquad \cos\angle ABC = \frac{\text{hyp}}{\text{opp}} =$$

$$\cos\angle ABC = \frac{\text{adj}}{\text{hyp}} = \frac{4}{5} \qquad \sec\angle ABC = \frac{\text{hyp}}{\text{adj}} =$$

$$\tan\angle ABC = \frac{\text{opp}}{\text{adj}} = \frac{3}{4} \qquad \cot\angle ABC = \frac{\text{adj}}{\text{opp}} =$$

Example 5: Choose the statement that is true.

A. $\sin^2 x + 1 = \cos^2 x$

B. $(\cos x)(\tan x) = \sec x$

C. $\sin x\left(1 + \cot^2 x\right) = \sin x$

D. $\frac{(\sin x)(\csc x)}{\cos x} = \sec x$

To solve this problem, you need to evaluate each of the statements to see if the left side of the statement can be rewritten to equal the right side.

Evaluate answer choice A:

A). $\sin^2 x + 1 = \cos^2 x$

Step A1: Choice A is false because the Pythagorean Identity states $\sin^2\theta + \cos^2\theta = 1$.

Evaluate answer choice B:

B). $(\cos x)(\tan x) = \sec x$

(1). $(\cos x)\left(\dfrac{\sin x}{\cos x}\right) = \sec x$

(2). $\sin x = \sec x$

Step B1: Replace tan x with sin x/cos x, a reciprocal identity.
Step B2: The cos x terms divide out.

Since sin x does not equal sec x, choice B is false.

Evaluate answer choice C:

C). $\sin x\left(1+\cot^2 x\right) = \sin x$

(1) $\sin x\left(\csc^2 x\right) = \sin x$

(2) $\sin x\left(\dfrac{1}{\sin^2 x}\right) = \sin x$

(3) $\sin x\left(\dfrac{1}{(\sin x)(\sin x)}\right) = \sin x$

(4) $\dfrac{1}{\sin x} = \sin x$

(5) $\csc x = \sin x$

Step C1:
Replace $\left(1+\cot^2 x\right)$ with $\left(\csc^2 x\right)$, a Pythagorean Identity.
Step C2:
Replace $\left(\csc^2 x\right)$ with $\left(\dfrac{1}{\sin^2 x}\right)$, a reciprocal identity.
Step C3:
Expand $(\sin^2 x)$ to $(\sin x)(\sin x)$. Step C4:
One sin x on the bottom of the fraction cancels the sin x on the top of the fraction, leaving
1/sin x.
Step C5: Replace (1/sin x) with (csc x), a reciprocal identity.

Since csc x does not equal sin x, choice C is false.

Evaluate answer choice D:

D). $\dfrac{\sin x(\csc x)}{\cos x} = \sec x$

(1) $\dfrac{\sin x\left(\frac{1}{\sin x}\right)}{\cos x} = \sec x$

(2) $\dfrac{1}{\cos x} = \sec x$

(3) $\sec x = \sec x$

Step D1: Replace csc x with (1/sin x), a reciprocal identity.
Step D2: The sin x terms in the numerator of Step D1 divide out to get 1.
Step D3: Replace (1/cos x) with sec x, a reciprocal identity.

Since sec x equals sec x, choice D is true.

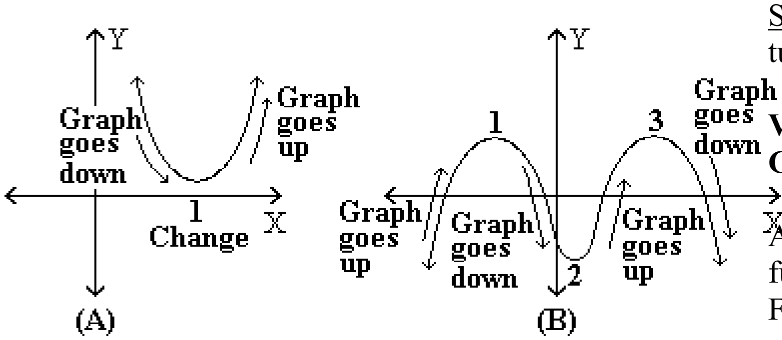
The correct answer is choice D.

Graphing Functions
A function is a relation between two variables such that each value of the first variable corresponds to exactly one value of the second variable. A polynomial function is a function that involves a series of terms added together. A polynomial function can be expressed in standard form as $f(x) = a_n x^n + a_{n-1} + \ldots + a_1 x + a_1$ where n represents the degree of the polynomial. (Remember, when you are working with functions f(x) and y can be interchanged.) The degree of a function is the largest exponent on the variable.

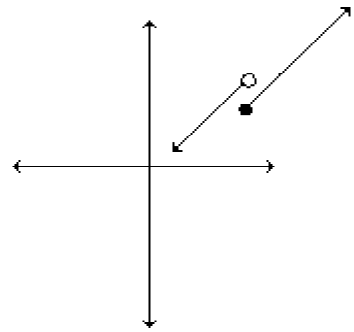
Graphs of Polynomial Functions:

The graph of a polynomial function has at most (n - 1) turns, that is, it can change direction at most (n - 1) times. The direction a graph is going (up or down) is always read from the left to the right. Graph A in the

diagram below changes direction once; whereas, graph B changes direction three times.



The graph of a continuous function will have no breaks. The graphs in the diagram above are graphs of continuous functions. A graph of a function that is not continuous is shown below.



All polynomial functions are continuous functions.

Example 1: State the maximum number of turns in the graph of the polynomial.

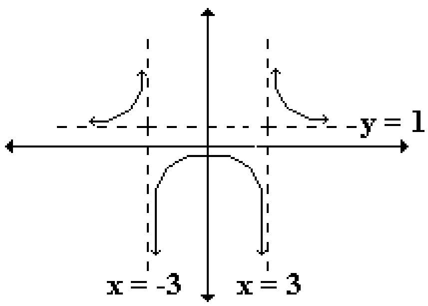
$$2x^5 + 3x^4 - 2x^3 + 3x^2 + 2x - 1$$

(1) n = 5
(2) 5 - 1 =4

Step 1: The degree of the polynomial is 5. The highest exponent on the x is 5.
Step 2: n - 1 = 4 so the maximum number of turns is 4.

Vertical and Horizontal Asymptotes of Graphs:

An asymptote is a line that the graph of a function comes close to, but never touches. For a rational function (a function with a numerator and a denominator), f(x) = g(x)/h(x), where g(x) and h(x) have no common factors and h(x) does not equal zero. It is possible to have vertical and/or horizontal asymptotes in rational functions. In the diagram below, the line y = 1 is a horizontal asymptote and the lines x = -3 and x = 3 are vertical asymptotes. Asymptotes are represented by dotted lines because points on the asymptotes are not points that lie on the graph of the function.



The vertical asymptotes are the zero(s) of h(x).
The horizontal asymptotes are as follows:

- The line y = 0 is the horizontal asymptote if the degree of g(x) < degree of h(x).
- The line y = a/b is the horizontal asymptote, where a is the leading coefficient of g(x) and b is the leading coefficient of h(x) if the degree of g(x) = degree of h(x).
- There is no horizontal asymptote if the degree of g(x) > degree of h(x).

A <u>coefficient</u> is the number that is multiplied by the variable(s) in a term. The <u>leading coefficient of a polynomial</u> is the coefficient of the first term when the polynomial is written in standard form (remember, the standard form for a polynomial has the exponents in descending order).

Example 2:

Name the asymptote(s) of the graph of the function, $f(x) = \frac{5x^2}{x-1}$

Find the vertical asymptotes first. Note that $g(x) = 5x^2$ and $h(x) = x - 1$.

(1) x - 1 = 0
(2) x = 1

<u>Step 1</u>: Solve for the zeros of h(x) by setting the denominator equal to zero.
<u>Step 2</u>: Add 1 to both sides of the equation to solve for x. The vertical asymptote is at x = 1.

Find the horizontal asymptotes.

(1) The degree of the numerator is 2 and the degree of the denominator is 1.
(2) Since the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote.

The function f(x) has only one asymptote. It is a vertical asymptote at x = 1.

Example 3:

Name the asymptote(s) of the graph of the fun

Find the vertical asymptotes first. Note that $g(x) = 3x^2$ and $h(x) = x^2 - 4$.

(1)	**(2)**	**(3)**
$x^2 - 4 = 0$	$x^2 - 4 = 0$	$\sqrt{x^2} = \pm\sqrt{4}$
	$+4 \quad +4$	$x = \pm 2$
	$x^2 = 4$	$x = +2\ ,\ x = \cdot$

<u>Step 1</u>: Solve for the zeros by setting the denominator equal to zero.
<u>Step 2</u>: Add 4 to both sides of the equation to solve for x.
<u>Step 3</u>: Take the square root of both sides of the equation to solve for x. Since 4 has two square roots, 2 and -2, there are two vertical asymptotes, x = 2 and x = -2.

Find the horizontal asymptotes.

(1) The degree of the numerator is 2 and the degree of the denominator is 2.
(2) The degree of the numerator is equal to the degree of the denominator. The leading coefficient of the numerator is 3 (so, a = 3) and the leading coefficient of the denominator is 1 (so, b = 1). Therefore, the horizontal asymptote is the line y = a/b = 3/1 or y = 3.

Answer: The function f(x) has three asymptotes: x = 2, x = -2, and y = 3.

Example 4:

Name the asymptote(s) of the graph of the fun

Find the vertical asymptotes first.

(1) $x - 3 = 0$
(2) $x = 3$

Step 1: Solve for the zeros by setting the denominator equal to zero.
Step 2: Add 3 to both sides of the equation to solve for x. The vertical asymptote is $x = 3$.

Find the horizontal asymptotes.

(1) The degree of the numerator is 0 (because the numerator is a constant) and the degree of the denominator is 1.
(2) Since the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is the line $y = 0$.

Answer: The function f(x) has two asymptotes: $x = 3$ and $y = 0$.

Right and Left Behaviors of a Graph:

There are four basic rules for determining the right and left hand behaviors of a graph of a function. Those rules are listed below.

- If the leading coefficient of a polynomial function is positive then the graph rises to the right.
- If the leading coefficient of a polynomial function is negative then the graph rises to the left.
- If the degree of the function is even, then the graph has the same right and left behavior.
- If the degree of the function is odd, then the graph has opposite right and left behaviors.

Example 5: Describe the left and right behaviors of the graph of $f(x) = 4x^3$.
The leading coefficient is positive, and the degree is 3, an odd number. The graph will rise to the right and fall to the left.

Example 6: Describe the left and right behaviors of the graph of $f(x) = 3x^2 + 2x + 1$.
(1) The leading coefficient is positive, so the graph will rise to the right.
(2) The degree is 2 (an even number), so the graph has the same right and left behavior. This means that the graph will also rise to the left.

The graph will rise to the right and rise to the left.

Horizontal and Vertical Shifts of Graphs:

When certain changes to an equation of a function are made, the graph of the new function may vary by moving the original graph to a new location in the coordinate plane, while the basic shape of the graph remains the same. The horizontal and vertical shifts of a graph caused by changes made to an equation can be summarized as follows.

If f(x) is a function:
- the graph of f(x) + a will result in a vertical shift upward of |a| units
- the graph of f(x) - a will result in a vertical shift downward of |a| units
- the graph of f(x + b) will result in a horizontal shift to the left |b| units
- the graph of f(x - b) will result in a horizontal shift to the right |b| units
- the graph of -f(x) will result in a reflection in the x-axis

A change to an equation can result in both a horizontal and vertical shift such as the graph of f(x + b) + a which will result in a vertical shift up |a| units and a horizontal shift to the left |b| units.

Example 7:
Compare the graph of $g(x) = x^2 + 9$ to the grap

Since the shift follows the f(x) + a formula, where a = 9, the graph will shift 9 units up.

Example 8:

Compare the graph of $g(x) = (x+3)^2 - 4$ to the graph of $f(x) = x^2$.

Since the shift follows the f(x + b) - a formula, where a = -4 and b = 3, the graph will shift 4 units down and 3 units to the left.

Example 9:
Compare the graph of $g(x) = -(x-2)^2 + 1$ to the graph of $f(x) = x^2$

Since the shift follows the -f(x - b) + a formula, where a = 1 and b = -2, the graph will reflect across the x-axis, shift 1 unit up and 2 units to the right.

Example 10: Compare the graph of g(x) = |x| - 3 to the graph of f(x) = |x|.

Since the shift follows the f(x) - a formula, where a = -3, the graph will shift 3 units down.

Example 11: Compare the graph of g(x) = |x + 1| + 4 to the graph of f(x) = |x|.

Since the shift follows the f(x + b) + a formula, where a = 4 and b = 1, the graph will shift 4 units up and 1 unit to the left.

Rational Expressions: Add/Subtract
A rational expression is a fraction whose numerator and denominator are polynomials. To add or subtract a rational expression, the denominators must be the same. If the denominators are the same, add or subtract the numerators and keep the common denominator.

Example 1: Subtract the fractions.

$$\frac{3}{8x} - \frac{5}{8x}$$

(1)	**(2)**	**(3)**
$\frac{3-5}{8x}$	$\frac{-2}{8x}$	$\frac{-2 \div 2}{8x \div 2} = \frac{-1}{4x}$

Step 1: Rewrite the expression as one fraction. The numerators will be subtracted and the denominator will remain the same.
Step 2: Subtract the numerators. 3 - 5 = -2
Step 3: Since 2 and 8 can both be divided by 2, the fraction can be reduced to -1/4x.

Example 2: Add the fractions.

$$\frac{2x}{3(x+2)} + \frac{5x+4}{3(x+2)}$$

(1)	**(2)**
$\frac{2x+5x+4}{3(x+2)}$	$\frac{7x+4}{3(x+2)}$

Step 1: Rewrite the expression as one fraction. The numerators will be added and the denominator will remain the same.
Step 2: Add the like terms, 2x and 5x, to get 7x. It is not necessary to make any changes to the denominator.

Example 3: $\frac{8t}{(t^2+8)^3} - \frac{14}{(t^2+8)^3} + \frac{4t-8}{(t^2+8)^3}$

(1)	**(2)**	**(3)**
$\frac{8t-14+4t-8}{(t^2+8)^3}$	$\frac{(8t+4t)+(-14-8)}{(t^2+8)^3}$	$\frac{12t-2}{(t^2+8)}$

Step 1: Rewrite the expression as one fraction. The denominator is the same on all three fractions, so it will remain the same.
Step 2: Collect the like terms together.
Step 3: Add 8t and 4t together to get 12t, and -14 and -8 together to get -22. It is not necessary to make any changes to the denominator.

To add or subtract rational expressions with

unlike denominators, find the lowest common denominator of the expressions. Then rewrite each expression as an equivalent rational expression using the lowest common denominator. Once the denominators are like, add or subtract the numerators and keep the common denominator.

To find the lowest common denominator, find the least common multiple of the denominators. Begin by factoring each denominator, if possible. Then multiply the individual factors together to determine the lowest common denominator of the fractions.

Example 4: Add the fractions.

$$\frac{17x^2}{18x^2}+\frac{5x+2}{3x^2}$$

(1)

$$3x^2 = 3 \cdot x^2$$
$$18x^2 = 6 \cdot 3 \cdot x^2$$

(2)

lowest common denominator

$$3 \cdot x^2 \cdot 6 = 18x^2$$

(3)

$$\frac{17x^2}{18x^2}+\frac{(5x+2)(6)}{3x^2(6)}$$

(4)

$$\frac{17x^2}{18x^2}+\frac{30x+12}{18x^2}$$

(5)

$$\frac{17x^2+30x+12}{18x^2}$$

Step 1: Determine the common denominator. To do this, list the denominator of each fraction, and break them down into their factors.
Step 2: Multiply the factors that the denominators have in common: 3 and x^2. We now need to multiply by 6 because it is the only factor left that has not been multiplied. The lowest common denominator is $18x^2$. Step 3: Rewrite the fractions with the common denominator. The fraction on the left already has the lowest common denominator as its denominator, so we do not need to change it, but we do need to change the fraction on the right. We multiply the numerator and denominator by 6 to get $18x^2$. Step 4: Use the distributive property to multiply (5x + 2) by 6. This involves multiplying 5x by 6 to get 30x and multiplying 2 by 6 to get 12.

Step 5: Rewrite the fractions as one fraction. Since there are no like terms in the numerator of the fraction, the numerator remains as it is and the denominator does not change.

Example 5: Add the fractions.

$$\frac{3x^2}{4x^5y^6}+7$$

(1)

$$\frac{3x^2}{4x^5y^6}+\frac{7}{1}\cdot\frac{4x^5y^6}{4x^5y^6}$$

(2)

$$\frac{3x^2}{4x^5y^6}+\frac{28x^5y^6}{4x^5y^6}$$

$$3x^2$$

(4)

$$\frac{x^2\left(3+28x^3y^6\right)}{4x^5y^6}$$

(5)

$$\frac{3+28x^3y^6}{4x^3y^6}$$

Step 1: Determine the common denominator. Since the whole number 7 has a denominator of 1, the common denominator of these two fractions is $4x^5y^6$. Rewrite 7/1 as a fraction with the common denominator. This involves multiplying the numerator and denominator of the fraction by $4x^5y^6$. Step 2: Multiply to make the fractions have common denominators. $7 \cdot 4x^5y^6 = 28x^5y^6$ Step 3: Add the numerators.
Step 4: Since the two terms in the numerator can both be divided by x-squared, factor x-squared out of the terms.
Step 5: Simplify the fraction by cancelling out the x-squared in the numerator and two

of the x's from the x to the fifth power in the denominator.

Example 6: Subtract the fractions.

$$\frac{m}{m+3} - \frac{m^2+3m}{m^2+5m+6}$$

(1) $$\frac{m^2+5m+6}{(m+3)(m+2)}$$

(2) $$\frac{m}{m+3} - \frac{m^2+3m}{(m+3)(m+2)}$$

(3) $$\frac{m}{m+3} \cdot \frac{m+2}{m+2} - \ldots$$

(4) $$\frac{m^2+2m}{(m+3)(m+2)} - \frac{m^2+3m}{(m+3)(m+2)}$$

(5) $$\frac{m^2+2m-(m^2+3m)}{(m+3)(m+2)}$$

(6) $$\frac{m^2+2m-m^2-3m}{(m+3)(m+2)}$$

(7) $$\frac{m^2-m^2+2m-3m}{(m+3)(m+2)}$$

(8) $$\frac{-1m}{(m+3)(m+2)}$$

Step 1: Factor the denominator of the second fraction. The factors of the denominator are (m + 3) and (m + 2).
Step 2: Rewrite the fractions with the second denominator in factored form.
Step 3: Determine the common denominator. Since each denominator has (m + 3) as a factor, the common denominator is (m + 3)(m + 2). Multiply the numerator and denominator of the first fraction by (m + 2) to rewrite the fraction with the common denominator. The second fraction does not need to be changed because it already has the common denominator.
Step 4: Distribute m to (m + 2). This involves multiplying each term in (m + 2) by m. Now the fractions have common denominators.
Step 5: Rewrite the fractions as one fraction. Remember to place the numerator of the second fraction in parentheses because the entire numerator is being subtracted from the numerator of the first fraction.
Step 6: Distribute the subtraction symbol to each term in the parentheses. This involves changing the sign of each term in the parentheses.
Step 7: Collect the like terms in the numerator.
Step 8: $m^2 - m^2 = 0$ and $2m - 3m = -1m$.

Example 7: Add the fractions.

$$\frac{2x}{5x-20} + \frac{x+8}{3x^2-12x}$$

(1) $$5x-20 = 5(x-4) \qquad 3x^2-12x = 3x(x-4)$$

$$\frac{2x}{5(x-\ldots}$$

(3) $$\frac{2x}{5(x-4)} \cdot \frac{3x}{3x} + \frac{x+8}{3x(x-4)} \cdot \frac{5}{5}$$

(4) $$\frac{6x^2}{15x(x-4)} + \ldots$$

Step 1: Factor each denominator. Each term in the denominator of the first fraction can be divided by 5, so 5 is factored out of the expression. Each term in the denominator of the second fraction can be divided by 3x, so 3x is factored out of the expression.
Step 2: Rewrite the fractions with the denominators factored.
Step 3: Each denominator has a factor of (x - 4), so that is automatically part of the denominator. The first fraction also has a factor of 5, so the numerator and denominator of the second fraction must be multiplied by 5 to acquire the common denominator. The denominator of the second fraction also has a factor of 3x, so the numerator and denominator of the first fraction must be multiplied by 3x to acquire the common denominator. The common denominator of these two fractions is 15x(x - 4).
Step 4: Perform the necessary multiplications. Now the fractions have common denominators and can be added together.
Step 5: Rewrite the fractions as one fraction. The numerators are added together. There are no like terms in the numerator, so the numerator cannot be simplified.

Example 8: Subtract the fractions.

$$\frac{6x}{16x^2-25} - \frac{9}{4x-5}$$

(1)

$$16x^2 - 25$$
$$(4x-5)(4x+5)$$

(2)

$$\frac{6x}{(4x-5)(4x+5)} - \frac{9}{4x-5}$$

(3)

$$\frac{6x}{(4x-5)(4x+5)} - \frac{9}{4x-5} \cdot \frac{4x+5}{4x+5}$$

(4)

$$\frac{6x}{(4x-5)(4x+5)} - \frac{36x}{(4x-5}$$

(5)

$$\frac{6x-(36x+45)}{(4x-5)(4x+5)}$$

(6)

$$\frac{6x-36x-45}{(4x-5)(4x+5)}$$

(7)

$$\frac{-30x-45}{(4x-5)(4x+}$$

Step 1: Factor the denominators. Since the denominator of the second fraction (4x - 5) cannot be factored, it is left alone. The denominator of the first fraction can be factored using the "difference of squares" rule. The factors are (4x - 5) and (4x + 5).
Step 2: Rewrite the fractions with the factored denominators.
Step 3: Determine the common denominator. Each fraction has (4x - 5) as a factor in the denominator, so (4x - 5) is part of the common denominator. The denominator of the first fraction also has (4x + 5) as a factor, so the numerator and denominator of the second fraction need to be multiplied by (4x + 5) to acquire the common denominator. The common denominator is (4x - 5)(4x + 5).
Step 4: Perform the necessary multiplications. Now the fractions have common denominators and can be subtracted.
Step 5: Rewrite the fractions as one fraction. Remember to place the numerator of the second fraction in parentheses because the entire numerator is to be subtracted from 6x.
Step 6: Distribute the subtraction symbol through the parentheses. This involves changing the sign of each number in the parentheses.
Step 7: Collect the like terms together and simplify. Since 6x and -36x are like terms, they can be added together to get -30x.

Example 9: Add the fractions.

$$\frac{2}{x^2-3x+2} + \frac{6x}{x^2-2x}$$

(1)

$$x^2-3x+2 \qquad x^2-2x$$
$$(x-1)(x-2) \qquad x(x-2)$$

$$\frac{2}{(x-1)(x-2)} \cdot \frac{x}{x}$$

common denominator

$$x(x-1)(x-2)$$

(3)

$$\frac{2x}{x(x-1)(x-2)} + \frac{6x^2-6x}{x(x-1)(x-2)} \qquad \frac{2x+}{x(x}$$

(5)

$$\frac{6x^2-4x}{x(x-1)(x-2)}$$

(6)

$$\frac{2x(3x-2)}{x(x-1)(x-2)}$$

Step 1: Factor the denominators. Then use the factors to determine the common denominator. The common denominator is x(x - 1)(x - 2).
Step 2: Multiply the numerator and denominator of the first fraction by x and the numerator and denominator of the second fraction by (x - 1).
Step 3: Perform the necessary multiplications. Now the fractions have common denominators and can be added.
Step 4: Rewrite the fractions as one fraction.
Step 5: Collect like terms.
Step 6: Since both terms in the numerator can be divided by 2x, we can factor 2x out of the numerator.
Step 7: The x (from 2x) in the numerator of the fraction and the x in the denominator of the fraction divide out to simplify the fraction.

Rational Functions: Multiply/Divide

A rational expression is a fraction whose numerator and denominator are polynomials. A rational expression is in its simplest form when the numerator and denominator have no common factors.

Simplifying Fractions:

If a, b, and c are nonzero real numbers, then $\frac{ac}{bc} = \frac{a}{b}$.
You can only divide out factors, not terms

that are being added or subtracted. For example, $\frac{x+1}{x}$ cannot be simplified because the x in the numerator is being added to the 1, so it is not a factor. Since the x in the numerator is not a factor, it cannot divide out with the x in the denominator.

To simplify a rational expression, factor the numerator and denominator. Then divide out any common factors.

Factoring:

There are four types of factoring. These types are listed below with an example of each type.

1. Factoring out the Greatest Common Monomial Factor:

$$4x^2 + 8x = 4x(x + 2)$$

$4x^2$ and $8x$ can both be divided by $4x$, so $4x$ is factored out

2. Factoring the Difference of Squares:

$$(9x^2 - 16) = (3x - 4)(3x + 4)$$

This factorization works because:

3x times 3x equals $9x^2$,
-4 times 4 equals -16
-12x plus 12x equals 0x.

3. Factoring a Perfect Square Trinomial:

$$x^2 + 20x + 100 = (x + 10)(x + 10) = (x + 10)^2$$

This factorization works because:

x times x equals x^2,
10 times 10 equals 100
10x plus 10x equals 20x.

4. Factoring a Trinomial:

$$x^2 - 9x + 14 = (x - 7)(x - 2)$$

This factorization works because:

x times x equals x^2,
-7 times -2 equals +14
-7x plus -2x equals -9x.

Remember to always factor out the greatest common monomial factor first, if there is one.

Example 1: Factor the following polynomial.

$$3x^2 + 36x + 60$$

(1)

$$\frac{3x^2}{3} + \frac{36x}{3} + \frac{60}{3}$$

$$3(x^2 + 12x + 20)$$

(2)

$$3(x^2 + 12x + 20)$$

$$3(x + 2)(x + 10)$$

Step 1: The three terms in the polynomial can be divided by 3, so we need to factor 3 out of the polynomial. Then write the new polynomial.
Step 2: Factor the polynomial that remains in the parentheses. The factors of the polynomial in parentheses are (x + 2) and (x + 10). The polynomial is now completely factored.

Answer: 3(x + 2)(x + 10)

Example 2: Simplify.

$$\frac{x^2 - 5x + 6}{x^2 - 3x}$$

(1)

$$x^2 - 5x + 6$$

$$(x - 3)(x - 2)$$

$$\frac{x^2}{x} - \frac{3x}{x}$$

$$x(x - 3)$$

(2)

$$\frac{(x - 3)(x - 2)}{x(x - 3)}$$

Step 1: Factor the numerator and the denominator. The three terms in the numerator do not have a common factor. The numerator must be factored using the factoring a trinomial method. The two terms in the denominator have a common factor of x. Divide x out of both terms to factor the denominator completely.
Step 2: Rewrite the expression with the numerators and denominators in factored form.
Step 3: The (x - 3) in the numerator and the (x - 3) in the denominator divide out to simplify the expression. The expression is now completely simplified.

Answer: (x - 2)/x

Multiplying Rational Functions:

To multiply rational expressions, multiply the numerators, multiply the denominators, and then simplify.

Example 3: Multiply and simplify.

$$\frac{2x^2}{14xy^2} \cdot \frac{10y^5}{6xy}$$

(1) $\frac{(2x^2)(10y^5)}{(14xy^2)(6xy)}$ **(2)** $\frac{20x^2y^5}{84x^2y^3}$ **(3)** $\frac{5y^2}{21}$

Step 1: Rewrite the fractions as one fraction. The numerators should be multiplied and the denominators should be multiplied.
Step 2: Multiply the numerators and the denominators. Remember that when multiplying numbers taken to powers that have the same bases, the exponents are added, so $x \cdot x = x^{1+1} = x^2$ and $y^2 \cdot y = y^{2+1} = y^3$. Step 3: Divide out common factors. 20 and 84 can both be divided by 4. The x^2 in the numerator and denominator divide out. When two terms with the same base are divided, their exponents are subtracted, so $\frac{y^5}{y^3} = y^{5-3} = y^2$.

Example 4: Multiply and simplify.

$$\frac{x^2+6x+9}{x^2+x} \cdot \frac{x+1}{x+3}$$

(1) $\frac{x^2+6x+9}{(x+3)(x+3)}$ $\frac{x^2}{x} + \frac{x}{x}$ $x(x+1)$ **(2)** $\frac{(x+3)(x+3)}{x(x+1)} \cdot \frac{x+1}{x+3}$

Step 1: Factor the numerators and denominators, if possible. x^2+6x+9 is a perfect square trinomial and has factors of (x + 3)(x + 3). Each of the terms of x^2+x has a factor of x, so the expression factors as x(x + 1). Neither the numerator nor the denominator of the second fraction can be factored, so they are left as they are.
Step 2: Rewrite the fractions with the factors in the correct places.
Step 3: Rewrite the fractions as one fraction. Remember to place the numerator and denominator of the second fraction in parentheses since they are to be multiplied.
Step 4: Divide out common factors. One of the (x + 3) terms in the numerator and the (x + 3) term in the denominator divide out. The (x + 1) term in the numerator divides out with the (x + 1) term in the denominator. The fraction is now completely simplified.

Example 5: Multiply and simplify.

$$\frac{(x+9)}{(x-7)} \cdot \frac{(x-7)}{1}$$

(1) $\frac{(x+9)}{(x-7)} \cdot \frac{(x-7)}{1}$ **(2)** $\frac{(x+9)(x-7)}{(x-7)}$ **(3)** $x+9$

Step 1: Rewrite the term (x - 7) as a fraction with 1 as the denominator and enclose all terms in parentheses.
Step 2: Multiply the numerators and the denominators.
Step 3: Divide out common factors.

Dividing Rational Expressions:

To divide one rational expression by another, multiply the first expression by the reciprocal of the second expression.

The reciprocal of a non-zero number $\frac{a}{b}$ is $\frac{b}{a}$.

Example 6: Divide and simplify.

$$\frac{x+1}{x^2+4x+4} \div \frac{x^2+8x+7}{x+2}$$

(1) $$\frac{x+1}{x^2+4x+4} \cdot \frac{x+2}{x^2+8x+7}$$

(2) $$\frac{x+1}{(x+2)(x+2)} \cdot \frac{x+2}{(x+1)(x+7)}$$

(3) $$\frac{(x+1)(x+2)}{(x+2)(x+2)(x+1)(x+7)}$$

(4) $$\frac{1}{(x+2)(x+7)}$$

Step 1: Rewrite the problem by multiplying the first expression by the reciprocal of the second expression.
Step 2: Factor the numerators and denominators, if possible. Neither of the numerators can be factored. The denominator of the first expression is a perfect square trinomial that factors as (x + 2)(x + 2). The denominator of the second expression is a factorable trinomial that factors as (x + 1)(x + 7).
Step 3: Multiply the numerators and the denominators. Remember to place the numerators in parentheses because they are complete terms.
Step 4: Divide out common factors. When all factors of the numerator divide out, the numerator is 1.

Example 7: Divide and simplify.

$$(x^2+8x) \div \frac{3x+24}{12x}$$

(1) $$\frac{(x^2+8x)}{1} \div \frac{3x+24}{12x}$$

(2) $$\frac{(x^2+8x)}{1} \cdot \frac{12x}{3x+24}$$

(4) $$\frac{x(x+8)(12x)}{3(x+8)}$$

(5) $$\frac{12x^2}{3}$$

Step 1: Rewrite the first expression as a fraction with 1 as the denominator.
Step 2: Rewrite the problem by multiplying the first expression by the reciprocal of the second.
Step 3: Factor the numerators and denominators, if possible.
Step 4: Multiply the numerators and the denominators. Remember to place the 12x in parentheses.
Step 5: Divide out common factors.
Step 6: Since 12 and 3 can both be divided by 3, we can still simplify. 12 ÷ 3 = 4 and 3 ÷ 3 = 1. Now the expression is completely simplified.

Trigonometric Ratios - B

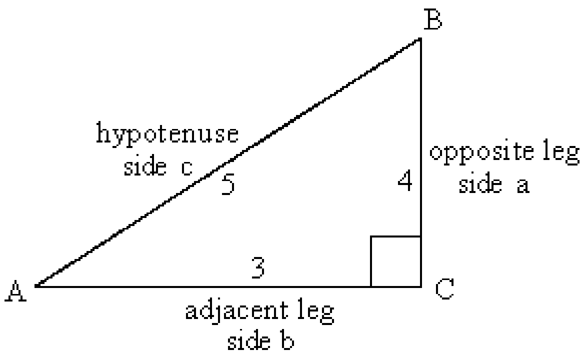
Trigonometry is the study of triangle measurement. A trigonometric ratio involves the ratio of the lengths of the sides of a right triangle.

Trigonometric ratios allow you to find the lengths of the sides of right triangles using

one of their acute angles. Acute angles are angles which measure less than 90 degrees. A right triangle has one angle which measures 90 degrees. The triangle side opposite the 90 degree angle is called the hypotenuse. The hypotenuse is the longest side of a right triangle.

Utilize this information to draw a right triangle. Label the 90 degree angle, Angle C. Label the opposite side, the hypotenuse, Side C. Label the acute angle above or below Angle C, Angle B. Label the side across from Angle B, Side B. Label the remaining angle, Angle A. Label the side opposite Angle A, Side A. Now assign measures: Side C = 5, Side B = 3, Side A = 4.

Your triangle should look like the triangle below.



The formulas for trigonometric ratios are:

<u>Sine</u> of an acute angle = measure of the opposite leg/measure of the hypotenuse

<u>Cosine</u> of an acute angle = measure of the adjacent leg/measure of the hypotenuse

<u>Tangent</u> of an acute angle = measure of the opposite leg/measure of the adjacent leg

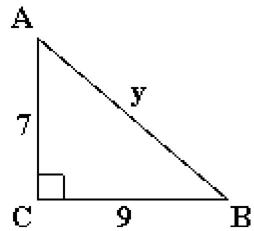
Therefore, the trigonometric ratios for angle A of the triangle you've drawn are:

sin A = 4/5
cos A = 3/5
tan A = 4/3

Students should be familiar with the Pythagorean Theorem which often needs to be applied in order to complete the third side of a right triangle prior to finding trigonometric ratios. The theorem asserts that in a right triangle the square of the hypotenuse is equal to the sum of the squares of the legs.

$$a^2 + b^2 = c^2$$

Example 1: Triangle ABC is a right triangle. Find the cosine of angle A. Rationalize the denominator.



(1)

$$a^2 + b^2 = c^2$$
$$(7)^2 + (9)^2 = c^2$$
$$49 + 81 = c^2$$
$$130 = c^2$$
$$c = \sqrt{130}\text{, so } y = \sqrt{130}$$

(2)

$$\cos(A) = \frac{\text{adj}}{\text{hyp}} = \frac{7}{\sqrt{13}}$$

<u>Step 1</u>: Determine the length of side y. Apply the Pythagorean Theorem.
<u>Step 2</u>: Substitute the values into the cosine formula.
<u>Step 3</u>: Rationalize the denominator.

Answer: $\cos(A) = \frac{7\sqrt{130}}{130}$

Trigonometric Tables

<u>Trigonometric tables</u> show the values of sine, cosine, and tangent for a large list of angle measures.

Trigonometric ratios depend on the measures of acute angles of right triangles. Acute angles are angles which measure less than 90 degrees. A right triangle has one

angle which measures 90 degrees. The triangle side opposite the 90 degree angle is called the hypotenuse. The hypotenuse is the longest side of a right triangle.

To read a trigonometric table, you need to know either the measure of an acute angle, or the sine (sin), cosine (cos), or tangent (tan) of the angle. The following table shows a portion of the table of trigonometric ratios.

Table of Trigonometric Ratios

Angle	Sin	Cos	Tan
12°	0.2079	0.9781	0.2126
13°	0.2250	0.9744	0.2309
14°	0.2419	0.9703	0.2493

Practice interpreting the table by finding the cosine of a 13° angle. The cosine of a 13° angle is 0.9744. Or, find the degree of an angle that has a tangent of 0.2493. The angle degree is 14.